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EUCLIDEAN METRIC INVARIANTS OF CONICS BY TENSOR ALGEBRA

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1. Introduction. This paper originated in a desire to furnish an elementary illustration of the scope and methods of tensor algebra to non-specialists in the field. Because of the familiarity of the transformation theory of the conics, this together with the formation of their euclidean metric invariants by tensor algebra has been chosen as the illustrative example. This paper will also serve as a link between the Aronhold-Clebsch symbolism of classical algebraic invariant theory and the modern tensor notation.

2. Coordinate transformations. The vanishing of the quadratic form

$$f_0 = (ax)^2 = (a_0x^0 + a_1x^1 + a_2x^2)^2 = g_{00}x^0x^0 + g_{11}x^1x^1 + g_{22}x^2x^2 + 2g_{01}x^0x^1 + 2g_{02}x^0x^2 + 2g_{12}x^1x^2$$

constitutes the equation in homogeneous coordinates x^A, $A=0, 1, 2$, of a general conic section. The expression $(ax)^2$ is purely symbolic, the a_A having no numerical significance when written alone but assuming the meaning $a_Aa_B=g_{AB}$ when written as a quadratic product. This notation was adopted by Aronhold and Clebsch, and most of the literature pertaining to algebraic invariant theory prior to the advent of tensor algebra employed this symbolism.

The projective geometry of the conic C, $f_0=0$, consists of those properties of C which appear the same to an observer in any one of the totality of coordinate systems $\bar{x}^A$ related to the system x^A by the projective transformations

$$x^A = \sum_0^2 a_Q^A \bar{x}^Q = a_Q^A \bar{x}^Q, \qquad \left| a_B^A \right| \neq 0,$$

and their inverses

$$\bar{x}^A = A_Q^A x^Q, \qquad a_Q^A A_B^Q = \delta_B^A, \qquad \delta_B^A = 0, \quad A \neq B, \qquad \delta_B^A = 1, \quad A = B.$$

Let $L=u_Qx^Q$ be a linear form whose vanishing gives the equation of the line L. Under $x\to\bar{x}$,

$$L = u_Q a_A^Q \bar{x}^A = \bar{u}_A \bar{x}^A = \bar{L},$$

so that $\bar{u}_A = u_Q a_A^Q$. Transforming f_0 in a similar way we obtain the transformation law of the coefficients g_{AB} of C,

$$x^A = a_Q^A \bar{x}^Q, \qquad \bar{u}_A = u_Q a_A^Q, \qquad \bar{g}_{AB} = g_{QR} a_A^Q a_B^R,$$
$$\bar{x}^A = A_Q^A x^Q, \qquad u_A = \bar{u}_Q A_A^Q, \qquad g_{AB} = \bar{g}_{QR} A_A^Q A_B^R.$$

We define as a *contravariant vector* λ^A, a *covariant vector* μ_A, and a *covariant tensor* μ_{AB} any quantity transforming cogrediently with x^A, u_A and g_{AB} respectively.

By an extension of this idea we are led to tensors of any rank. For example, if both ϵ^{ABC} and ϵ_{ABC} have the values $+1$ and -1 in *all coordinate systems* according as the indices form an even or odd permutation of the natural order 0, 1, 2, and the value 0 otherwise, then

$$|a|\epsilon^{ABC} = \bar{\epsilon}^{QRS}a_Q^A a_R^B a_S^C, \qquad \bar{\epsilon}_{ABC} = |a|^{-1}\epsilon_{QRS}a_A^Q a_B^R a_C^S,$$

and ϵ^{ABC} and ϵ_{ABC} are the components of contravariant and covariant tensors of rank three. But furthermore the components of these tensors are the *same in all coordinate systems* and for this reason they are called *invariant* tensors. The presence of the determinental factor $|a|$ is signified by calling ϵ^{ABC} and ϵ_{ABC} *relative* tensors of *weights* $+1$ and -1 respectively.

The contraction of a covariant vector μ_A with a contravariant vector λ^A gives

$$\mu_Q\lambda^Q = \bar{\mu}_R A_S^R a_Q^S \bar{\lambda}^Q = \bar{\mu}_Q\bar{\lambda}^Q$$

so that an invariant form results from this process. All invariants of tensor algebra may be formed by this method of contracting a covariant with a contravariant index.

3. Projective invariants of the conic. We display in this section the projective invariants of a point x^A, a line u_A, and the conic g_{AB}. These will arise as the simultaneous invariants of the three ground forms

$$(3.1)\qquad f_0 = (aX)^2 = (bX)^2 = (cX)^2, \quad f_1 = (uX) = u_Q X^Q, \quad F_1 = (xU) = x^Q U_Q,$$

where for the moment X^A and U_A represent general point and line coordinates.

By the first fundamental theorem of the symbolic method, Clebsch[1] has shown that all the projective invariants of the forms (3.1) which are integral and rational in the coefficients of the forms are expressible in terms of the symbolic factors (ax), (ux), (abc), and (abu). The latter are formed by all possible contractions of a covariant with a contravariant linear symbol and by all possible determinants of exclusively covariant or exclusively contravariant linear symbols. It is well known that there are but four basic invariants of the system (3.1), namely

$$D = (abc)^2 = 6|g_{AB}|, \qquad f_0 = (ax)^2 = g_{QR}x^Q x^R, \qquad J_0 = (ux) = u_Q x^Q,$$
$$(3.2)\qquad f'' = (abu)^2 = |g_{AB}|g^{QR}u_Q u_R, \qquad (g_{BQ}g^{QA} = \delta_B^A),$$

and that these constitute a *complete basic system of invariants* of the forms (3.1) in the sense that any rational integral invariant of (3.1) is expressible in terms of these four.

The invariance of $|g_{AB}|$ follows at once from $\bar{g}_{AB} = g_{QR}a_A^Q a_B^R$ on forming the determinant of both sides, $|\bar{g}_{AB}| = |a|^2|g_{AB}|$. This same invariant is also expressible as a result of solving the last of (3.2) for g^{AB},

$$2g^{AB} = |g_{AB}|^{-1}\epsilon^{AQR}\epsilon^{BST}g_{QS}g_{RT}$$

and contracting with g_{AB} to obtain

$$6|g_{AB}| = \epsilon^{AQR}\epsilon^{BST}g_{AB}g_{QS}g_{RT}.$$

4. The euclidean metric group. The projective group of transformations a_B^A may be restricted to become the euclidean metric group E of rotations and translations. This is done by insisting that the circular points $(0; 1, \pm i)$ defined as the singular line conic

$$\delta^{AB}u_Au_B = u_1^2 + u_2^2 = (0u_0 + u_1 + iu_2)(0u_0 + u_1 - iu_2) = 0$$

have the same coordinates to within a factor of proportionality in all systems, and that the coordinates of any point $(x^0; x^1, x^2)$ not on the line at infinity, L_∞, with the equation $\delta_Q^0x^Q = x^0 = 0$, have the coordinates $(1; x^1, x^2)$.

The resulting transformation equations are $x^A = a_Q^A\bar{x}^Q$, where

$$\begin{Vmatrix} a_0^0 & a_1^0 & a_2^0 \\ a_0^1 & a_1^1 & a_2^1 \\ a_0^2 & a_1^2 & a_2^2 \end{Vmatrix} = \begin{Vmatrix} 1 & 0 & 0 \\ a & \cos\theta & -\sin\theta \\ b & \sin\theta & \cos\theta \end{Vmatrix}.$$

Thus E is determined as a subgroup of the projective group by insisting upon the formal invariance of $\phi = (lX) = X^0$ and $\Phi = (\tau U)^2 = U_1^2 + U_2^2$ in all coordinate systems. The transformation equations of g_{AB} under the two subgroups of E are

(4.1)

(translation)	(rotation)
$\bar{g}_{00} = g_{00} + g_{rs}a_0^ra_0^s$	$\bar{g}_{00} = g_{00}$
$\bar{g}_{0i} = g_{0i} + g_{ri}a_0^r$	$\bar{g}_{0i} = g_{0s}a_i^s$
$\bar{g}_{ij} = g_{ij}$	$\bar{g}_{ij} = g_{rs}a_i^ra_j^s.$

5. The euclidean metric ground forms. E is determined as a subgroup of the projective group by stipulating that the forms $\phi = X^0$ and $\Phi = U_1^2 + U_2^2$ remain unaltered. By an application of the adjunction principle of algebraic invariant theory a complete basic system of invariants of a point x^A, a line u_A, and a conic g_{AB} under E will be found by adjoining the two forms ϕ and Φ to the forms (3.1),

(5.1) $\quad f_0 = (aX)^2, \quad f_1 = (uX), \quad \phi = (lX), \quad F_1 = (xU), \quad \Phi = (\tau U)^2 = (\nu U)^2,$

and *asking for a complete basic system of invariants of these forms under the projective group.*

By the same application of the first fundamental theorem of the symbolic method as introduced in section 3, all the projective invariants of the augmented system of forms (5.1) may be built from the symbolic factors

$(ax), \quad (ux), \quad (lx), \quad (a\tau), \quad (u\tau), \quad (abc), \quad (abu), \quad (abl), \quad (aul), \quad (x\tau\nu).$

But τ and ν may occur only in quadratic combinations of the type

$$(a\tau)(b\tau) = a_1b_1 + a_2b_2 = (a \mid b)$$

and

$$(x\tau\nu)(a\tau)(b\nu) = (lx)(abl);$$

and hence all the projective invariants of (5.1) may be built from the symbolic factors

(5.2) (ax), (ux), (lx), $(a \mid b)$, $(a \mid a)$, $(u \mid u)$, (abc), (abu), (abl), (aul).

It is clear that any projective invariant of (5.1) will be *a fortiori* an invariant of the subgroup E, but it is not obvious that conversely any invariant of (3.1) under E is likewise a *projective* invariant of the augmented system (5.1). R. Weitzenböck[2] has proved that this converse theorem holds and hence that any invariant of (3.1) under E may be built from the symbolic factors (5.2).

6. Tensor algebra and its geometrical significance. The coefficients of the forms (5.1), g_{AB}, u_A, δ^0_A, x^A and δ^{AB}, plus the alternating invariant tensor ϵ^{ABC} of section 2 will form the building blocks of the tensor formulation of the invariants analogous to the symbolic factors (5.2). That these tensors suffice for the formation of all the projective invariants of (5.1) is a basic theorem due to C. M. Cramlet.[3] In the tensor algebra of invariants it plays the rôle of the first fundamental theorem of the symbolic method cited in section 3. We now make certain observations which will be needed later.

a. The transformation law of the line coordinates u_A decomposes into $\bar{u}_0 = u_0 + u_r a^r_0$ and $\bar{u}_i = u_r a^r_i$ so that the vanishing of the coordinates u_i has a significance independent of the coordinate system, that is, the direction u_i is then that of L_∞.

b. $g^{AQ}u_Q$ is the pole of u_A as to C. If this pole is likewise on u_A then $g^{AQ}u_Q u_A = 0$ and u_A is a tangent of C.

c. Dually, $g_{AQ}x^Q$ is the polar of x^A as to C. Let y^A be on this polar, then $g_{AQ}x^Q y^A = 0$. But $g_{AQ}y^Q$ is the polar of y^A and hence if y^A is on the polar of x^A, so likewise is x^A on the polar of y^A.

d. $\delta^0_Q \epsilon^{QBC} = \epsilon^{0BC}$ is an invariant tensor giving the contravariant coordinates of the line δ^0_A, namely L_∞. $\epsilon^{0QR}u_Q v_R = \epsilon^{0qr}u_q v_r = u_1v_2 - u_2v_1 = 0$ is the condition for intersection of the lines u_A and v_A on L_∞, that is, for parallelism of the directions u_i and v_i. Since only the components ϵ^{0ik} are operative in contractions, the simpler notation $\epsilon^{ik} = \epsilon^{0ik}$ will be employed in such contractions.

e. $\epsilon^{0AQ}u_Q$ is the intersection point of the line u_A with L_∞.

f. $\delta^{RS}u_R v_S = \delta^{rs}u_r v_s = u_1v_1 + u_2v_2 = 0$ is the condition for orthogonality of the directions u_i and v_i. If $v_A = u_A$ the line u_A is self-orthogonal and the directions u_i form two parallel pencils on the circular points $(0; 1, \pm i)$. Again, since only the components δ^{ij} are effective in contraction, only these components will be indicated in contractions.

g. A pencil of parallel lines with the direction u_i will intersect L_∞ in the point $\epsilon^{0As}u_s$. The direction v_j of the polar of this point is given by $v_j = g_{jr}\epsilon^{rs}u_s$. The directions u_j and v_j are said to be *conjugate* with respect to C. When C is a central conic $|g_{lm}| \neq 0$, the conjugate relationship is reciprocal and this reciprocity may be brought into evidence by introducing the factor $(-|g_{lm}|)^{-1/2}$ so that

$$v_j = (-|g_{lm}|)^{-1/2} g_{jr}\epsilon^{rs}u_s$$

and

$$v_j = (-|g_{lm}|)^{-1} g_{jr}\epsilon^{rs}g_{si}\epsilon^{ti}v_i,$$

since

$$(-|g_{lm}|)^{-1} g_{jr}\epsilon^{rs}g_{si}\epsilon^{ti} = \delta^t_j.$$

The condition that the direction conjugate to u_i be at the same time orthogonal to u_i is

$$\delta^{jr}g_{rs}\epsilon^{sk}u_j u_k = 0. \tag{6.1}$$

h. $\delta^{AQ}u_Q$ is a point on L_∞ since $\delta^{0A}=0$. If v_A is any line on this point, $v_A\delta^{AQ}u_Q=0$ and the directions v_i and u_i are orthogonal. Hence $\delta^{AQ}u_Q$ is the point on L_∞ determined by the direction orthogonal to u_i.

i. $\epsilon^{ABQ}u_Q$ are the contravariant components of the line u_A.

j. $y^A = \epsilon^{ABS}u_R v_S$ is the point of intersection of the lines u_A and v_A, for $u_A y^A = v_A y^A = 0$.

7. The euclidean metric invariants of the conic. The complete basic system of invariants of the forms (5.1) in terms of the symbolic factors (5.2) has been found by R. Weitzenböck.[4] We tabulate these in both the symbolic notation used by Weitzenböck and the tensor notation of this paper. Any rational integral invariant of g_{AB}, x^A, and u_A under E is expressible in terms of these 18 basic invariants.

The verification that no one of the 18 basic invariants, say S_1, is expressible rationally and integrally in terms of the remaining 17 invariants of the system may be made in the following way. From a consideration of the degrees 2, 1, 1 of S_1 in g_{AB}, x^A, and u_A respectively it is apparent that the condition for dependency of S_1 is that there exist constants $k_0, \cdots, k_4$, wherein $k_0 \neq 0$ and at least one of the remaining k's is not zero, for which

$$k_0 S_1 + k_1 L\Omega + k_2 C J_0 + J_1(k_3 V_2 + k_4 T_2) = 0$$

holds *identically* in g_{AB}, x^A and u_A. But for the selection

$$u_0 = u_1 = x^1 = x^2 = g_{11} = g_{22} = g_{01} = 0$$

it is clear that $J_0 = J_1 = \Omega = 0$, while $S_1 = -g_{02}g_{12}u_2x^0 \neq 0$ and hence $k_0 = 0$ and S_1 is independent.

Name	Type	Symbolic form	Tensor form	Degree in g_{AB}	Degree in x^A	Degree in u_A
D	g_{AB} only	$(abc)^2$	$6\lvert g_{AB}\rvert$	3	0	0
J_1		$(a\lvert a)$	$\delta^{rs}g_{rs}$	1	0	0
C		$(abl)^2$	$2\lvert g_{ij}\rvert$	2	0	0
f_0	g_{AB} and x^A	$(ax)^2$	$g_{QR}x^Qx^R$	1	2	0
L		(lx)	x^0	0	1	0
W		$(xa)(a\lvert b)(bx)$	$g_{Qr}\delta^{rs}g_{sR}x^Qx^R$	2	2	0
S_2		$(abl)(ax)(b\lvert c)(cx)$	$g_{As}\epsilon^{sr}g_{rt}\delta^{tu}g_{uB}x^Ax^B$	3	2	0
f''	g_{AB} and u_A	$(abu)^2$	$2\lvert g_{AB}\rvert g^{QR}u_Qu_R$	2	0	2
Φ		$(u\lvert u)$	$\delta^{rs}u_ru_s$	0	0	2
V_1		$(a\lvert u)^2$	$g_{rs}\delta^{ri}\delta^{sj}u_iu_j$	1	0	2
T_1		$(aul)(a\lvert u)$	$g_{rs}\delta^{ri}\epsilon^{sj}u_iu_j$	1	0	2
Ω		$(abu)(abl)$	$g_{Au}\epsilon^{uv}g_{vB}\epsilon^{ABQ}u_Q$	2	0	1
J_0	g_{AB}, x^A and u_A	(ux)	u_Qx^Q	0	1	1
V_2		$(xa)(a\lvert u)$	$g_{As}\delta^{sr}u_rx^A$	1	1	1
T_2		$(aul)(ax)$	$g_{As}\epsilon^{sr}u_rx^A$	1	1	1
R_1		$(abu)(ax)(b\lvert u)$	$g_{AQ}g_{Br}\delta^{rs}u_s\epsilon^{QBC}u_Cx^A$	2	1	2
R_2		$(abu)(ax)(b\lvert c)(cx)$	$g_{Ar}\delta^{rs}g_{sQ}\epsilon^{QRC}u_Rg_{CB}x^Ax^B$	3	2	1
S_1		$(abl)(ax)(b\lvert u)$	$g_{Ar}\epsilon^{rs}g_{st}\delta^{tu}u_ux^A$	2	1	1

8. Reduction to canonical forms. Before interpreting geometrically the vanishing of the basic invariants the equation of the proper conics for which $\lvert g_{AB}\rvert \neq 0$ will be reduced to its canonical forms. L_∞ intersects C in the pair of points given by $g_{rs}x^rx^s=0$. Whence $C=2\lvert g_{ij}\rvert<0$, $C=0$ and $C>0$ this pair will be real and distinct, real and equal, and conjugate complex so that C will be a hyperbola, parabola, or an ellipse respectively. If L_∞ is to cut C in the circular points $(0;\ 1,\ \pm i)$ then $g_{12}=0$ and $g_{11}-g_{22}=0$. Thus, for real coefficients, the condition that an ellipse be a circle is that

$$J_1^2 - 2C = (g_{11} - g_{22})^2 + 4g_{12}^2 = 0.$$

We first reduce to canonical form the equation of the central conics for which $\lvert g_{jk}\rvert \neq 0$. Under the translation $x^i=\bar{x}^i+a_0^i$ it may be seen from (4.1) that $\bar{g}_{0i}=g_{0i}+g_{qi}a_0^q$. Thus the axes may be translated to a new origin at the point a_0^A for which $\bar{g}_{0i}=0$. Now the center y^A of C is characterized by the condition that the polar of the center is L_∞, $g_{QA}y^Q=k\delta_A^0$, so that $g_{0i}+g_{qi}y^q=0$ and the new origin at a_0^A is the center of C. By a normalization of the homogeneous coefficients we may choose $k=-1$. In the new system the center y^A is $(1;\ 0,\ 0)$ so that $g_{0i}=0$ and $g_{00}=-1$. Under any subsequent rotation without translation it may also be seen from (4.1) that the relations $g_{0i}=0$ and $g_{00}=-1$ are not disturbed.

A further reduction of the equation of the central conic may be effected by a rotation of axes about its center to bring them into coincidence with the unique

pair of orthogonal conjugate diameters whose directions must be the roots of the quadratic equation $\delta^{ir}g_{rs}\epsilon^{sk}u_iu_k=0$ as given by (6.1). But with the coordinate axes now in these directions the equation must admit the solutions $(0;1,0)$ and $(0;0,1)$ so that $g_{12}=0$. We thus arrive at the familiar canonical form,

$$-(x^0)^2+g_{11}(x^1)^2+g_{22}(x^2)^2=0,$$

which is an ellipse or an hyperbola according as $|g_{jk}|=g_{11}g_{22}>0$ or <0 respectively.

In the case of the parabola for which $|g_{jk}|=0$, let λ^A be a point on L_∞ with the polar $g_{Ar}\lambda^r$. If this polar is to be L_∞ itself, $g_{ir}\lambda^r=0$, so that the non-trivial solution of this pair of homogeneous equations gives the point of tangency of C with L_∞. If x^A be any point not on L_∞, its polar $g_{AQ}x^Q$ will have the direction $g_{i0}+g_{ir}x^r$. The parallel pencil on λ^A will have the direction $\epsilon_{ir}\lambda^r$ and the condition that the polar of x^A be orthogonal to this direction is

$$\delta^{ij}\epsilon_{ir}\lambda^r(g_{j0}+g_{js}x^s)=0.$$

For variable x^i this is the equation of the axis of the parabola. If we choose the coordinate system to make this axis $x^2=0$, then $\lambda^1\neq 0$, $\lambda^2=0$ and $g_{ir}\lambda^r=0$ gives $g_{11}=0$, $g_{12}=0$, and the equation of the axis becomes $-\lambda^1(g_{02}+g_{22}x^2)=0$, from whence $g_{02}=0$ and $g_{22}\neq 0$. The axis will cut C in the points given by $x^0(2g_{01}x^1+g_{00}x^0)=0$, and on choosing the origin $(1;0,0)$ to be at one of these intersections, $g_{00}=0$ and $g_{01}\neq 0$. Normalization by division by g_{22} gives the canonical form $(x^2)^2+2g_{01}x^0x^1=0$.

9. Geometrical interpretation of the invariants. In the same paper in which he tabulated the 18 invariants of a conic, a point, and a line, Weitzenböck likewise partially interpreted geometrically the vanishing of each. To point out the ease with which the geometric meaning of an invariant equation may be read from its tensor formulation we shall interpret in much greater detail than did Weitzenböck the vanishing of the two most interesting invariants, R_1 and R_2. Section 6 should be consulted as the key to these interpretations.

$R_1=(\epsilon^{QBC}u_C)(g_{QA}g_{Br})(x^A\cdot\delta^{rs}u_s)=0$ for given u_A reads: (contravariant line u_A) (has a covariant pole) (which is collinear with x^A and the point on L_∞ in the direction normal to u_i). Thus $R_1=0$ gives the line through the pole of u_A which is orthogonal to u_A. R_1 vanishes identically in x^A for u_A the L_∞, since the line joining the center of C with any point x^A in the plane is orthogonal to L_∞.

$R_1=(x^Ag_{AQ}\epsilon^{QBC})(u_C\cdot g_{Br}\delta^{rs}u_s)=0$ for given x^A reads: (contravariant polar of x^A) (is incident with intersection Q of line u_A and diameter of C conjugate to the direction normal to u_i). Thus to construct the line conic $R_1=0$ corresponding to a given point x^A, choose Q any point on the polar of x^A and pass a diameter D through Q. Then the line u_A passes through Q orthogonal to the direction conjugate to D.

For the central conic,

$$-(x^0)^2 + g_{11}(x^1)^2 + g_{22}(x^2)^2 = 0, \qquad 1/g_{11} - 1/g_{22} = c^2 \geqq 0,$$

and for the parabola $(x^2)^2+2g_{01}x^0x^1=0$,

$$R_1 = g_{11}g_{22}(-x^2u_0u_1 + x^1u_0u_2 + c^2x^0u_1u_2) = 0,$$
$$R_1 = g_{01}(x^0u_0u_2 + x^2u_1^2 - (g_{01}x^0 + x^1)u_1u_2) = 0.$$

If the conic is a circle, $R_1=0$ implies $(-x^2u_1+x^1u_2)u_0=0$ and the line conic degenerates into the pencil of diameters and the family of lines orthogonal to the diameter through x^A.

If the point x^A is on the major (transverse) axis, $x^2=0$, the line conic degenerates into the family of lines orthogonal to the major (transverse) axis and into a pencil of lines on the point $(1; c^2/x^1, 0)$. Thus $R_1=0$ defines on the major (transverse) axis an involution of points whose double points are the two foci. Similarly if x^A be taken on the minor (conjugate) axis, $x^1=0$, there is defined an involution of points $(1; 0, x^2)\rightarrow(1; 0, -c^2/x^2)$ whose double points are the pair of imaginary foci.

In the case of the parabola, if x^A is on L_∞, the line conic degenerates into the family $u_1=0$ of lines parallel to the axis and into the family of lines perpendicular to the direction $(0; x^1, x^2)$. If x^A is on the axis $x^2=0$ the line conic also degenerates, this time into the family of lines $u_2=0$ orthogonal to the axis and into the pencil on the point $(1; -g_{01}-x^1, 0)$. Thus $R_1=0$ sets up an involution on the axis of the parabola whose double points are the focus and the infinite point on the axis.

$R_2=(\epsilon^{QRC}u_R)(g_{Q*}g_{BC})(x^B)(\delta^{r*}g_{rA}x^A)=0$ for given u_A reads: (the contravariant line u_A) (has a covariant pole) (collinear with x^B) (and with the point on L_∞ perpendicular to the polar of x^A). Thus to construct the point x^A, let P be the pole of u_A and draw any line L through P. Draw the diameter of C conjugate to the direction orthogonal to L. This diameter will intersect L in the desired point x^A.

For the central conic and the parabola

$$R_2 = (g_{11}g_{22})^2(c^2u_0x^1x^2 + g_{11}^{-2}u_1x^0x^2 - g_{22}^{-2}u_2x^0x^1) = 0,$$
$$R_2 = -g_{01}(g_{01}^2u_2x^0x^0 + u_1x^1x^2 - (g_{01}u_1 + u_0)x^0x^2) = 0.$$

If C is a circle, the conic $R_2=0$ degenerates into L_∞ and the diameter perpendicular to u_i.

If the line u_A is perpendicular to the major (transverse) axis, so that $u_2=0$, then $(c^2u_0x^1+g_{11}^{-2}u_1x^0)x^2=0$ and $R_2=0$ degenerates into the major (transverse) axis and a line perpendicular to this axis so that to the line $(u_0; u_1, 0)$, or $x^1=-u_0/u_1=X$, there corresponds the line $\bar{x}^1=(1/X)(a^2/e^2)$, $a^2=1/g_{11}$ and $e^2=c^2/a^2$. The double lines of the involution are the two directices $x^1=\pm a/e$ of the conic. Similarly if the line u_A is perpendicular to the minor (conjugate) axis there is defined an involution of lines orthogonal to this axis whose double lines are the pair of imaginary directrices.

In the case of the parabola, if the line u_A is taken normal to the axis so that $u_2=0$, the conic

$$R_2 = -g_{01}(u_1x^1 - (g_{01}u_1 + u_0)x^0)x^2 = 0$$

degenerates into the axis and a line orthogonal to it. Thus $R_2=0$ sets up an involution among the normals to the axis in which to the line $(u_0;\ u_1,\ 0)$, or $x^1=-u_0/u_1=X$, there corresponds the line $x^1=g_{01}-X$, the double lines of the involution being the directrix and L_∞. If the line u_A is parallel to the axis, $u_1=0$, then $(g_{01}^2u_2x^0-u_0x^2)x^0=0$ and $R_2=0$ degenerates into L_∞ and a line parallel to the axis. Thus $R_2=0$ sets up an involution of lines parallel to the axis whereby the line $(u_0;\ 0,\ u_2)$, or $x^2=-u_0/u_2=Y$, has as its corresponding line $x^2=(-1/Y)g_{01}^2$ so that the double lines of the involution are $x^2=\pm ig_{01}$ which are the two polars of the circular points $(0;\ 1,\ \pm i)$.

$R_2=(g_{CB}x^B)(x^Ag_{Ar}\delta^{rs}g_{sQ})(\epsilon^{QRC})(u_R)=0$ for given x^A reads: (the polar of x^A) (and the diameter conjugate to the direction orthogonal to the polar of x^A) (intersect in a point) (which is the center of the pencil of lines u_A). For a central conic, R_2 vanishes identically in u_A when x^A is at the center. For a parabola, R_2 vanishes identically either when x^A is the point of contact of C with L_∞ or when x^A is the intersection with L_∞ of the tangent at the vertex.

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That famous motto. Plato's writings do not convince any mathematician that their author was strongly addicted to geometry. . . . We know that he encouraged mathematics. . . . But if—which nobody believes—the *μηδεὶς ἀγεωμέτρητος εἰσίτω* [Let no man ignorant of geometry enter] of Tzetzes had been written over his gate, it would no more have indicated the geometry within than a warning not to forget to bring a packet of sandwiches would now give promise of a good dinner.—Augustus De Morgan (1858).—*Contributed.*

A little known property of the normal probability curve. It is a little known but easily proved fact that the inflectional tangents of the normal probability curve strike the base line two sigma units each side of the axis of symmetry. This property of course holds equally well when the x- and y-units are unequal. Consequently, lines joining these base points with the inflection points should appear to be tangent to the curve. Checks of the diagrams in ten common texts fail to conform to this test.—*W. F. Cheney, Jr.*

LUCAS'S TESTS FOR MERSENNE NUMBERS

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1. The theorem. The purpose of this note is to give a brief self-contained proof of the following theorem, which includes as special cases the famous Lucas tests for primality of Mersenne numbers.

THEOREM. *Let $q=2^k-1$ and suppose integers c, d are known such that d and c^2-d are quadratic non-residues of q, if q is prime. Define $W_2=2(c^2+d)/(c^2-d)$, $W_{i+1}=W_i^2-2$. Then q is prime if and only if* $q \mid W_k$.*

Note. The conditions of the theorem are a little peculiar in that we are called upon to announce in advance that d and c^2-d would be non-residues of q, if q were prime; we then use this information to test the primality of q. However because of the special form of $q=2^k-1$, we can be certain that some numbers are non-residues if q is prime, *e.g.*, -1, -2, 3, 6, and products of these by squares. By trial we can then find a variety of suitable values of c and d. We list below all distinct values of W_2 obtainable with c^2 and $|d|$ less than 100.

c	d	W_2
1	-2	$-2/3$
1	3	-4
3	6	10
5	-2	46/27
9	6	58/25
5	27	-52

These six values of W_2 provide us with as many "universal" tests for Mersenne numbers, $W_2=-4$ being the one discovered by Lucas. If however the form of q is further prescribed, there may be still other valid choices of c and d. For example, if k is of the form $4n-1$, then 5 is a non-residue of q, and we may take $d=5$, $c=1$, $W_2=-3$. This is the other of Lucas's two tests and is the one he used to prove that $2^{127}-1$ is prime.

2. Preliminary lemmas. Our proof of the above theorem is modelled on one given by D. H. Lehmer† for the case $W_2=4$.

Define $U_n=(a^n-b^n)/(a-b)$ with $a=c+\sqrt{d}$, $b=c-\sqrt{d}$, and let $V_n=a^n+b^n=U_{2n}/U_n$. In the following lemmas it will always be tacitly assumed that the primes under discussion do not divide $2abcd$.

We begin by noting two readily verified identities:

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* If W_k is not an integer, we mean that q divides the numerator of W_k when the latter is in its lowest terms. In a particular case we could in any event always begin by making W_2 an integer mod q.

† Journal London Math. Soc. 10, 1935, 162–165. The author is indebted to Professor Lehmer for his suggestions concerning this paper.

$$U_{m+n} = U_nU_{m+1} - abU_mU_{n-1}, \tag{1}$$

$$U_n^2 - U_{n-1}U_{n+1} = (ab)^{n-1}. \tag{2}$$

LEMMA 1. *A prime cannot divide two successive U's.*

Proof. Take $n=2$ in (1):

$$U_{m+2} = (a+b)U_{m+1} - abU_m. \tag{3}$$

It follows that a prime which divides two successive U's will divide all of them, including $U_1=1$.

LEMMA 2. *A prime cannot divide both U_n and V_n.*

Proof. From (1) with $m=n$ we obtain

$$\begin{aligned} V_n &= U_{n+1} - abU_{n-1} \\ &= (a+b)U_n - 2abU_{n-1} \quad \text{by (3).} \end{aligned}$$

A prime which divides both U_n and V_n will therefore also divide U_{n-1}.

LEMMA 3. *The integers r for which a prime p divides U_r are all multiples of a single integer.*

Proof. It will suffice to show that $p \mid U_m$ and $p \mid U_n$ imply that $p \mid U_{n\pm m}$ $(n>m)$. That $p \mid U_{n+m}$ is evident from (1). Now replace n by $n-m$ in (1):

$$U_n = U_{n-m}U_{m+1} - abU_mU_{n-m-1}.$$

Then $p \mid U_{n-m}U_{m+1}$. But p cannot divide both U_m and U_{m+1}; hence $p \mid U_{n-m}$.

The next two lemmas form the foundation on which the Lucas theory is built. For completeness we include the familiar proofs.*

LEMMA 4. *If p is prime, $U_p \equiv (d \mid p) \pmod p$.*

Proof. From the definition of U_n we readily find

$$U_p = pc^{p-1} + \binom{p}{3}c^{p-3}d + \binom{p}{5}c^{p-5}d^2 + \cdots + d^{(p-1)/2}.$$

All the binomial coefficients except the last vanish mod p. Hence

$$U_p \equiv d^{(p-1)/2} \equiv (d \mid p) \pmod p.$$

LEMMA 5. *If p is prime, $U_{p-(d|p)} \equiv 0 \pmod p$.*

Proof.

$$U_{p+1} = (p+1)c^p + \binom{p+1}{3}c^{p-2}d + \cdots + (p+1)cd^{(p-1)/2}.$$

* See for example Hardy and Wright, An Introduction to the Theory of Numbers, pp. 147–149.

Mod p, the only binomial coefficients which survive are the first and last. Also $c^p \equiv c \pmod{p}$ by Fermat's "little theorem." Hence

$$U_{p+1} \equiv c(1 + d^{(p-1)/2}) \pmod{p} \tag{4}$$
$$\equiv 0 \pmod{p} \quad \text{if and only if} \quad (d \mid p) = -1.$$

Now set $n = p$ in (2). We find, using Lemma 4 and Fermat's theorem,

$$U_{p-1}U_{p+1} \equiv 0 \pmod{p}. \tag{5}$$

Lemma 5 follows at once from (4) and (5).

LEMMA 6. *If p is prime, and $(ab \mid p) = (d \mid p) = -1$, then $p \mid V_t$, where $t = (p+1)/2$.*

Proof. By Lemma 5, $p \mid U_{2t}$. Next take $n = t$, $m = t-1$ in (1).

$$U_p = U_t^2 - abU_{t-1}^2.$$

If $p \mid U_t$ we have

$$U_p \equiv (d \mid p) \equiv -1 \equiv -abU_{t-1}^2 \pmod{p},$$

in contradiction of $(ab \mid p) = -1$. Hence $p \nmid U_t$, $p \mid V_t = U_{2t}/U_t$.

LEMMA 7. *If $q \mid V_t$ with t a power of 2, and $q \leqq 4t^2 - 4t$, then q is prime.*

Proof. If q is not prime, select a prime divisor $p \leqq \sqrt{q}$. Then $p \mid V_t$ and a fortiori $p \mid U_{2t}$. The smallest integer r for which $p \mid U_r$ is then, by Lemma 3, a power of 2. But by Lemma 5, $r \leqq p+1$, and $p+1 \leqq \sqrt{q}+1 < 2t$. Hence $r \leqq t$ and p divides both U_t and V_t in contradiction of Lemma 2.

3. Proof of the theorem. The completion of the proof is now immediate. We note that $W_2 = (a^2+b^2)/ab = V_2/ab$ and an induction readily verifies that

$$W_k = V_{2^{k-1}}/(ab)^{2^{k-1}}.$$

Now suppose $q \mid W_k$, then $q \mid V_{2^{k-1}}$ a fortiori, and by Lemma 7 q is prime. Conversely if q is prime, the hypotheses of Lemma 6 are fulfilled and $q \mid V_{2^{k-1}}$. Since q is prime to $ab = c^2 - d$, we have $q \mid W_k$.

Saccheri's dilemma. Did [Saccheri] really believe that his beautiful Non-Euclidean geometry was nothing but a reductio-ad-absurdum? In order to guess Saccheri's own mind we have to remember, that in the close and constant intercourse with other mathematicians of his Order, his genius may have been intimidated by repeated discussions, and that before he could publish anything, the manuscript had to pass the censorship of at least two of the Jesuits.—G. B. Halsted (1910).
—*Contributed.*

Send suitable items to Professor E. T. Bell, California Institute of Technology, Pasadena 4, California.

A PROPERTY OF APPELL SETS*

C. J. THORNE, Louisiana State University

1. Introduction. An Appell set of polynomials $\{P_n(x)\}$, $n=0, 1, \cdots$ ($P_n(x)$ of degree n) is a set for which

$$P_n'(x) = P_{n-1}(x), \qquad n = 1, 2, \cdots . \tag{1}$$

An equivalent definition is the existence of a formal power series $A(t)=\sum a_n t^n$ ($a_0 \neq 0$) such that (formally)

$$A(t)e^{tx} = \sum P_n(x)t^n. \tag{2}$$

All summations are from 0 to ∞ unless otherwise stated. $A(t)$ is called the generating function of the polynomials. Each such $A(t)$ defines an Appell set. Some well known sets are $(x-a)^n/n!$, the Hermitian polynomials, the Bernoulli polynomials, and the Euler polynomials.

Sheffer† studied such sets. He proved that an Appell set satisfies a linear differential equation of the form

$$\sum L_r(x)y^{[r]}(x) = \lambda y(x), \tag{3}$$

if and only if $A(t)=e^{Q(t)}$. Here summation is 0 to n, $L_r(x)$ is a polynomial in x of degree not exceeding r, λ is a parameter, $[r]$ means the rth derivative, and $Q(t)$ is a polynomial in t. Moreover, except for $Q(t)=$constant, the minimum order of such differential equations is the same as the degree of $Q(t)$. Webster‡ has proven that any Appell set which is at the same time an orthogonal set in the usual sense is reducible to the Hermitian set by a linear transformation.

The purpose of this paper is to give a characterization of Appell polynomials in terms of a Stieltjes integral. Appell polynomials are then more useful in applications of the functional method.§

2. A new characterization. Let $\alpha(x)$ be a function of bounded variation on‖ (a, b) for which the integrals

$$\mu_n = \int_a^b x^n d\alpha(x), \qquad n = 0, 1, \cdots \tag{4}$$

all exist. Further assume that

* The present paper grew out of work the author did while a fellow at Brown University during the summer of 1942.

† Summary paper, I. M. Sheffer, Some applications of certain polynomial classes. Am. Math. Soc. Bulletin, December 1941, 885–898.

‡ M. S. Webster, Orthogonal polynomials with orthogonal derivatives. Am. Math. Soc. Bulletin, December 1938, 880–888.

§ C. J. Thorne and J. V. Atanasoff, A functional method for the solution of thin plate problems applied to a square clamped plate with a central point load. Iowa State College Journal of Science, vol. XIV, 333–343.

‖ This can be an infinite interval.

$$\mu_0 \neq 0. \tag{5}$$

Then

THEOREM I. *There exists a unique sequence of polynomials* $\{\phi_n(x)\}$, $n=0, 1, \cdots, \phi_n$ *being of degree* n, *such that*

$$\int_a^b \phi_n^{[r]}(x)d\alpha(x) = \delta_n^r = \begin{cases} 0, & n \neq r, \\ 1, & n = r. \end{cases} \tag{6}$$

Proof. $\phi_n(x)$ will be of the form

$$\phi_n(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0. \tag{7}$$

Relations (6) applied to (7) yield the results

$$\begin{aligned} &n!a_n\mu_0 = 1, \\ &\frac{n!}{1!}a_n\mu_1 + (n-1)!a_{n-1}\mu_0 = 0, \\ &\frac{n!}{2!}a_n\mu_2 + \frac{(n-1)!}{1!}a_{n-1}\mu_1 + (n-2)!a_{n-2}\mu_0 = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &a_n\mu_n + a_{n-1}\mu_{n-1} + a_{n-2}\mu_{n-2} + \cdots + a_0\mu_0 = 0. \end{aligned} \tag{8}$$

The system (8) has a unique solution if

$$n!(n-1)!(n-2)!\cdots 1\mu_0^{n+1} \neq 0. \tag{9}$$

Since $\mu_0 \neq 0$, we have the usual unique solution to (8):

$$a_{n-r+1} = \frac{\begin{vmatrix} \mu_0 & 0 & 0 & \cdots & 0 & 1 \\ \frac{\mu_1}{1!} & \mu_0 & 0 & \cdots & 0 & 0 \\ \frac{\mu_2}{2!} & \frac{\mu_1}{1!} & \mu_0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\mu_{r-1}}{(r-1)!} & \frac{\mu_{r-2}}{(r-2)!} & \frac{\mu_{r-3}}{(r-3)!} & \cdots & \frac{\mu_1}{1!} & 0 \end{vmatrix}}{(n-r+1)!\mu_0^r}. \tag{10}$$

THEOREM II. *The set* $\{\phi_n(x)\}$ *is an Appell set.*

Proof. In (6) change n to $n+1$ and r to $r+1$ and let $\psi_n(x) = \phi_{n+1}^{[1]}(x)$. Then

$$\int_a^b \psi_n^{[r]}(x)d\alpha(x) = \delta_{n+1}^{r+1} = \delta_n^r, \tag{11}$$

whence by uniqueness, $\psi_n = \phi_n$, *i.e.* $\phi_{n+1}^{[1]}(x) = \phi_n(x)$.

COROLLARY. *The generating function $A(t)$ for $\{\phi_n(x)\}$ is given by*

$$A(t) = \left[\int_a^b e^{tx} d\alpha(x)\right]^{-1} = \left[\sum \frac{t^m}{m!}\mu_m\right]^{-1}. \tag{12}$$

Proof. From definition (2),

$$A(t)e^{tx} = \sum t^n \phi_n(x). \tag{13}$$

By property (6),

$$A(t)t^r \int_a^b e^{tx} d\alpha(x) = t^r.$$

Hence

$$A(t) = \left[\int_a^b e^{tx} d\alpha(x)\right]^{-1} = \left[\sum \frac{t^m}{m!}\mu_m\right]^{-1}.$$

THEOREM III. *Every Appell set has the property* (6). *More precisely: To every Appell set $\{P_n(x)\}$ corresponds a function $\alpha(x)$ of bounded variation in* $(0, \infty)$ *such that*

(a) *the integrals* (4) *all exist on the range* $(0, \infty)$;

(b) $\mu_0 \neq 0$;

(c) $\int_0^\infty P_n^{[r]}(x) d\alpha(x) = \delta_n^r$.

Proof. Let $A(t) = \sum a_n t^n$ $(a_0 \neq 0)$ be the generating function for $\{P_n(x)\}$, and define $\{\mu_n\}$ by the formal relation $A(t) = [\sum t^m/m!\,\mu_m]^{-1}$. Boas* has shown that a function $\alpha(x)$ of bounded variation on $(0, \infty)$ exists (not however uniquely) such that

$$\mu_n = \int_0^\infty x^n d\alpha(x), \qquad n = 0, 1, \cdots.$$

Hence

$$A(t) = \left[\int_0^\infty e^{tx} d\alpha(x)\right]^{-1}, \tag{12'}$$

and by the corollary to Theorem II, relations (6) are seen to hold. Since $a_0 \neq 0$, therefore $\mu_0 \neq 0$.

* Widder, The Laplace Transform, p. 139.

THE NEED FOR COOPERATIVE ACTION IN MATHEMATICAL EDUCATION

RALEIGH SCHORLING, University of Michigan

In dealing with the problem of this symposium I shall, with the exception of a single suggestion, limit myself to the needs of teachers of mathematics in our elementary and secondary schools. Surely, I need not, to this group, emphasize the fact that this level of training is of paramount importance. If our institutions are to be not only surely preserved but also constantly improved, if the right kind of foundation for more advanced work is to be laid, the national organizations dealing with advanced mathematical training need to give careful attention to the mathematics of the earlier years.

How shall we improve the mathematical offering of the lower schools? Obviously, a necessary step, though not the complete solution, is to design better undergraduate mathematical curricula for the prospective teachers of mathematics for our elementary and secondary schools. And, how can we do this job on which we have all so grievously failed?

Fortunately we were given excellent advice forty-two years ago by Professor E. H. Moore in his presidential address before the American Mathematical Society. May I read you four quotations in which I need not change a single word in order to make them apply to today's crucial problems in mathematical education?

> "I wish . . . to express the hope that we may secure the active cooperation of many colleagues in the domains of science and of administration, so that the first carefully chosen steps of a really important advance movement may be taken in the near future."
>
> "Undoubtedly in many parts of the country improvements in organization and methods of instruction in mathematics have been made these last years. All persons who are, or may become, actively interested in this movement of reform should in some way unite themselves, in order that the plans and the experience, whether of success or failure, of one may be immediately made available in the guidance of his colleagues. . . ."
>
> "Teaching must become more of a profession. And this implies not only that the teacher must be better trained for this career but also in his career he be given greater freedom with greater responsibility. To this end closer relations should be established between the teachers of the colleges and those of the secondary schools; standing provisions should be made for conferences as to improvement of the secondary school curricula . . . , and the leading secondary school teachers should be steadily encouraged to devise and try out plans looking in any way toward improvement."
>
> "Furthermore, there is evident need of a national organization having its center of gravity in the whole body of science instructors in the secondary

schools; and those of us interested in these questions will naturally relate ourselves also to this organization."*

Many college teachers of mathematics feel, and justly so, that they have an important role in the education of teachers for the elementary and secondary schools. In national organizations dealing with higher education, there are individuals vitally concerned with the work in the lower schools. In fact, some of our national societies have standing committees on the education of teachers. There is, however, no machinery, no way by which the constructive thinking of the leaders in the various national organizations can be mobilized and be brought to bear continuously on common problems relating to pedagogy and to the undergraduate curricula for prospective teachers.

The teacher of mathematics in elementary and secondary schools *needs* your guidance and support. A few years ago mathematics in these grades was subjected to drastic criticisms. Incidentally, some were no doubt valid. The common practice of schools to allow the omission of all mathematics from the programs of pupils beyond the eighth grade was undoubtedly an over-simplification—at any rate it was devastating to the education of many students. Though the training program of the schools of the armed forces did not uncover this serious situation, it did publicize the shortage in mathematics.

At the moment, mathematics in the high school is enormously popular. There have been vast increases in the numbers that elect the various high school courses. There are good reasons for believing that this gratifying situation will continue for some time in the postwar period. The fact that the mental climate with respect to high school mathematics has changed is not due to the efforts of the American Mathematical Association and the National Council of Teachers of Mathematics. Time and circumstances have played in our favor, but they may not do so a second time. We need to keep a discerning eye on the situation that may exist five years after the war ends. Unless we provide sensible programs (note the plural) in mathematics for all pupils of the secondary schools, each appropriate to the ability and need to be served, the devastating fire of criticism may again be directed at mathematics. At any rate, it should be.

We need your help on a number of perplexing problems which in the long run affect your main interest. As illustrations, let us consider the following:

1. *The traditional sequential courses in high school mathematics will need to be further revised.* A fraction of the school's student body—unfortunately no one knows how large—will need a broader and deeper foundation in mathematics than educators and even teachers of mathematics have realized. No one should assume that all is well with the traditional courses. In most schools they are woefully out-of-date as regards both subject matter and method. Far too many men with good native ability came to the technical jobs of the army and navy with no clear understanding of such important concepts as vector, tolerance, interpolation, representative fraction, scale drawing, tangent of an angle, mi-

* Presidential address delivered before The American Mathematical Society at its ninth annual meeting, December 29, 1902. Science, N. S. Vol. XVII, pp. 401–416, March 13, 1903.

crometer, vernier, gage block, metric system, practical constructions, logarithms, and slide rule. Yet these are among the very things that would have added vitality and meaning to the abstract symbolism and theory which their former mathematics teachers struggled to teach them.

The traditional sequential courses should be reserved for the pupils who can profit by such courses and who want to take them. The door to the four-year sequence should be wide open, but no one should be pushed through it. Failure should, for the most part, be avoided by proper guidance to other appropriate mathematics courses. The unique values of the traditional courses cannot be achieved by a constant gearing down, nor by the inclusion of popular material in a futile effort to meet the mathematical needs of pupils who should not have elected the courses in the first place. The first step in the postwar period is to scrutinize the traditional courses in the light of recommendations that have long been widely approved.

2. *The necessary competency in dealing with whole numbers, common fractions, decimals, and percent must be attained.* The widely publicized letter by Admiral Nimitz* dramatized to the nation the alleged shortage in arithmetic. It would be futile to point out that this drastic indictment was based on wholly inadequate data collected by outmoded tests, for the reason that the main observation of low competency in arithmetic is amply supported by data that have been available to school people for more than a decade.

The simplicity of mathematical skills needed by the masses in the armed forces is even greater than was generally assumed, but these skills, nevertheless, are very difficult for many persons to achieve. Provision for growth in the mastery of arithmetic should be continuous throughout the elementary and secondary schools. Achievement of ability in mathematical reasoning and in sensible use of mathematical concepts requires much time and continuous practice. Courses of study of the elementary school should be reconsidered, and adequate time should be given to the teaching of arithmetic. Moreover, we have not given enough attention to arithmetic in teaching the senior high school courses. In fact, we have too often allowed the fundamental skills to deteriorate.

In the past two decades, professional courses for teachers dealing with the teaching of arithmetic practically disappeared from our teachers colleges, with the result that most beginning teachers do not know how to teach arithmetic. In fact, some actually fear it and escape the task by excessive attention to other activities in which they feel more confident. To make matters worse, many are tempted to teach the little they do teach by the incidental method—presumably the ideal method, but also the most difficult which only a few out of a hundred teachers could manage even if they had good training. The teaching of arithmetic can be and must be improved. Here is a problem that is not trivial, and one on which we need help.

3. *The secondary school should guarantee functional competence in mathematics to all who can possibly achieve it.* Competence in mathematics in many respects

* This MONTHLY, vol. 49, 1942, pp. 212–214.

parallels literacy in communication. The legal specification of literacy is fulfilled by the ability to write and to speak one's own language. In pioneer days, no doubt, this ability was adequate. Today the United States Army uses the phrase "functional literacy" to imply fourth-grade ability. Though no one knows exactly what functional literacy means, except by arbitrary definition, it is nevertheless clear that modern technology has stepped up the minimum requirements of literacy in communication.

In great-grandfather's day, when life was relatively simple, the ability to compute accurately when dealing with whole numbers, common fractions, decimals, and percent was adequate for the common affairs of life. There are good reasons for believing that the minimum requirement in mathematics for effective citizenship is moving up and is already *higher than mere control of the four fundamentals of arithmetic.*

The war has demonstrated in a dramatic way what is nonetheless true in peace time, but not so readily discernible; namely, that a boy lacking certain mathematical competencies is a pathetic victim of a ruthless system. In the armed forces men are so badly needed that they are expected to learn the impossible overnight, and we pour vast sums into their training; in peace times they shift from job to job and finally drift into the ranks of the unemployed.

An understanding and appreciation of the important elements of our culture, one of the admittedly valid aims of education, is contingent upon possession of certain basic concepts and principles from the areas of mathematics, biology, physics, chemistry, geology, and other sciences. These concepts and principles are the instruments without which man cannot think intelligently about machines, housing, business, transportation, communication, public and personal health, investments, insurance, or scientific inquiry. An understanding of these aspects of modern life demands a common language of quantitative symbols, fundamental operations, and basic ideas—perhaps some sixty in number—of arithmetic, of algebra, and of geometry.

Thus, functional competence in the mathematics of common affairs seems as crucial as functional literacy in communication. Unfortunately, it is not generally recognized that the minimum mathematical training needed for citizenship is higher than the level commonly achieved by the product of our schools. From the point of view of the educated person, the minimum mathematical equipment needed in common affairs is absurdly simple. But the task of insuring mathematical competence to all people is enormously difficult—one that has never been done in any community in all history. The job facing mathematics teachers is almost certain to be more difficult and more important in the years ahead. It is a challenge that mathematics teachers should accept. Here is a third problem on which we need your guidance.

4. *The mathematics departments of our secondary schools should provide appropriate courses for the neglected groups of students—very large ones—whose needs cannot possibly be met in traditional courses.* The secondary schools have had too many dissatisfied customers who, in depression days, accumulated as unemployed youths and in C.C.C. camps. The military training program has taken

these boys and has taught them a simple technical science and related mathematics. Several hundred thousand boys undoubtedly rendered outstanding service in saving our institutions by the technical science and mathematics they learned *after* leaving school. Moreover, a program correlating and emphasizing industrial arts, science, and mathematics is probably a good approach to the *general education* of many pupils who are never going to be satisfied with a purely academic program. Our secondary school has never been able to come to grips with the problem of mass education—one obstacle being that those concerned with higher education have not fully realized the dual responsibility of the high schools, first, for giving good training to future leaders, and second, for providing a general education appropriate for the major fraction of the high school population. This can be done. In fact, it must be done if in the long run our institutions are to survive. But it is enormously difficult without the full support of higher institutions.

We must provide a more realistic curriculum for the large number of persons who will continue to be absorbed by industry, trade, farm, and business fairly early in life. In particular, we must give more attention to the needs of industry which the traditional mathematics teacher has neglected far too long. Since there are in our high schools these large groups of students whose needs cannot possibly be met in traditional mathematical courses, the sensible thing to do is to provide good courses with very different goals and different experiences for groups with different needs. Furthermore, we must somehow do this in a manner that does not stigmatize any group. This task will be done better if the Mathematical Association of America provides help and guidance.

5. *The undergraduate curricula of prospective teachers of mathematics need to be improved.* The crucial manpower shortage as regards mathematics in the early months of the war suggests that the standard training of mathematics teachers may be inadequate. It seems that too many teachers lack familiarity with a vast range of uses of mathematics such as those, for example, in aviation, industry, trade, business statistics, and in general the simpler applications of a practical physics. It will be difficult for some to teach, not to mention design, courses having the new emphasis that the postwar era appears to demand. Perhaps the main difficulty is, as Professor Moore suggested, that there is too large a gap between pure and applied mathematics. Several institutions are providing courses for prospective teachers that include extensive work experiences in industry and business. It may be that the undergraduate curricula for prospective teachers of high school mathematics need to include a course that correlates a block of simple practical applications drawn from industrial arts, physics, and mathematics. I have no conviction on these guesses. My task is completed when I urge you to consider this problem and to decide what responsibility you have in helping us find the right answer. The fact that your program committee selected this topic for this symposium is evidence that some members in our Association believe that this problem requires your attention.

The foregoing is perhaps sufficient to show that high school teachers of

mathematics need your help. The next question is. Do they want your cooperation? The answer is clear and unequivocal. The American Mathematical Association has in the past thirty years been of great help to high school mathematics teachers. We have for example profited greatly by the friendly guidance of Herbert E. Slaught, the wise strategy of J. W. Young, the driving enthusiasm of Earl R. Hedrick, and the challenging vision of E. H. Moore. However, the complexity of our problems may now require an even higher degree of cooperation. The Association may need to spend some money if it wants a job done right. Research funds of the type that initiated and supported the Committee of 1923 are now hard to get. Some of the most competent members of our Association may need to lay aside for a time pure research and devote their energy to practical problems of the lower schools. Do not overlook the fact that J. W. Young gave three of his best professional years in order that he might render a service that thoughtful mathematics teachers of high school will deeply appreciate for generations to come.

If effective action can best be reached by the road of cooperation, then it seems desirable to select the groups that offer a hopeful beginning. I should like to outline a situation which invites active collaboration.

There is the Commission on Postwar Plans of the National Council of the Teachers of Mathematics. This Commission published its first report in the 1944 May number of the *Mathematics Teacher*. This Commission is a hard-working group of vigorous personalities who in normal times do not see eye to eye on many issues, but who have been driven by the exigencies of these grave times to splendid teamwork. The Commission has a number of projects under way which I believe are of vital interest to this Association. For example, the Commission recently proposed to the Board of Directors of the National Council of Teachers of Mathematics, that the Council be reorganized so as to consist of three sections, the first section to deal with the mathematics of the elementary school, the second, with the mathematics of the high school, and the third, with the mathematics of the junior college. You will agree, I think, that the mathematics of the junior college, which grew up like Topsy, is in need of a careful study. This undertaking might well be the joint effort of the Association and the Council.

I have been instructed, by unanimous vote, to invite the Association to work with the Commission on all matters that relate to the mathematics of the lower schools and, in particular, to join us in the study of the junior college offering in mathematics.

The second possibility for cooperation is provided by the Cooperative Committee on Science Teaching which represents five national organizations including the Mathematical Association of America. The Committee was created in 1941 and has been supported by funds from one of the foundations. It works on educational problems which no single scientific group can solve by working alone. Among these problems has been that of the certification and college preparation of science teachers for high schools. Another problem attacked by the Committee was that of using high school science and mathematics to meet

manpower needs during the war. A further function of this committee has been to serve as a forum in which representatives of the scientific societies have been able to state the views of their own groups and to learn the convictions of other groups on the teaching of science and mathematics at the elementary and secondary levels.

It is now proposed that the work of the Committee be reviewed and that it be reconstituted by the parent organizations. It is hoped that the Committee will be reorganized to include the following nine groups: physics, mathematics, chemistry, biology, research in science teaching, engineering, geology, astronomy, and physiography. Since the Committee has, at long last, made a good beginning in the manner suggested in one of the quotations from Professor E.H. Moore, I should like to urge that the Mathematical Association of America continue to participate in the work of that Committee.

The third opportunity for effective joint action is offered in cooperation with the United States Office of Education. Recently, Commissioner John W. Studebaker invited the National Council of Teachers of Mathematics to summon a small committee to consider ways and means by which the United States Office of Education might render more and better service to teachers of mathematics. We prepared a six-page report in which we listed things that the Office of Education might do for teachers of mathematics on three levels, in elementary, in secondary, and in higher institutions, to improve the teaching of mathematics. We were guided by the criteria that the Office of Education should not undertake anything unless (1) it is a problem national in scope, and (2) it cannot be undertaken by an individual, by a state, or an educational institution without excessive cost, duplication of effort, or the chance of getting incomplete and unreliable reports. Our report is now before the budget committee with the full support of Commissioner Studebaker. Newspapers have in recent days announced President Roosevelt's endorsement of the reorganization of the Office of Education that includes the services suggested in our report. The proposal is at the moment in a promising stage. However, anyone wise in the ways of Washington, especially of Congress, will not be too hopeful. Assuming that the proposal is rejected this year, one may venture the prediction that this project will go forward in the not too distant future, if it is given adequate support. With the exception of publishing the report by the National Committee on Mathematical requirements in 1923 and the Report on Preinduction Courses in Mathematics in 1943, the United States Office of Education has done little that has been of value to the cause of mathematics, although there are many services that long ago should have been provided for teachers of mathematics. It is to be hoped that the Mathematical Association of America may give this new proposal its hearty support.

With this long preface, I express the hope that the Mathematical Association of America may take the following steps:

1. To create a standing committee to give special attention to all vital ques-

tions relating to the mathematics of grades 1–14 inclusive, and to undergraduate curricula for the prospective teachers of mathematics for these grades.

2. To elect a member to the Commission on Postwar Plans of the National Council of the Teachers of Mathematics. (No one knows how long the Commission will need to operate, but the Mathematical Association of America can, of course, end its support at any time without notice.)

3. To participate in the work of the Cooperative Committee on Science Teaching. (A memorandum entitled "Proposal for the Continuance and Reconstitution of the Cooperative Committee on Science Teaching" has been submitted to the officers of the Mathematical Association of America.)

4. To vote its hearty approval of the proposal submitted by a committee of the National Council of Teachers of Mathematics to the U. S. Office of Education. (A document entitled "Functions of the Office of Education with Respect to the Teaching of Mathematics" has been submitted to the officers of the Mathematical Association of America.)

I now close with a fifth quotation from Professor Moore which, with only one word changed, seems appropriate:

> "I have accordingly felt at liberty to bring to the attention of the *Association* these matters of pedagogy of elementary mathematics, and I do so with the firm conviction that it would be possible for the *Association*, by giving still more attention to these matters, to further most effectively the highest interests of mathematics in this country."

A testimonial to Euclid. He studied and nearly mastered the six books of Euclid since he was a member of Congress.

He began a course of rigid mental discipline with the intent to improve his faculties, especially his powers of logic and language. Hence his fondness for Euclid, which he carried with him on the circuit till he could demonstrate with ease all the propositions in the six books; often studying far into the night, with a candle near his pillow, while his fellow-lawyers, half a dozen in a room, filled the air with interminable snoring.—Abraham Lincoln (Short Autobiography, 1860).

Another testimonial to Euclid. Geometry is nothing if it be not rigorous. . . . The methods of Euclid are, by almost universal consent, unexceptionable in point of rigor.—H. J. S. Smith (1873).

A final testimonial to Euclid. It has been customary when Euclid, considered as a textbook, is attacked for his verbosity or his obscurity or his pedantry, to defend him on the ground that his logical excellence is transcendent, and affords an invaluable training to the youthful powers of reasoning. This claim, however, vanishes on a close inspection. His definitions do not always define, his axioms are not always indemonstrable, his demonstrations require many axioms of which he is quite unconscious. A valid proof retains its demonstrative force when no figure is drawn, but very many of Euclid's earlier proofs fail before this test. . . . The value of his work as a masterpiece of logic has been very grossly exaggerated.—Bertrand Russell (1902).—*Contributed.*

A UNIFORM METHOD OF SOLVING CUBICS AND QUARTICS

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1. Introduction. While the literal solutions of algebraic equations of degree less than five are well-known, there seems to be a lack of a general method of attack which will solve them in a uniform manner. In this note such a general method is given, basing its theory on the properties of a cyclic matrix which occurs frequently in the discussion of alternating current symmetrical machines.

2. Outline of the method. Although the method does not work, beyond a certain point, for quintics and equations of still higher degree, we shall outline the procedure in terms of a quintic and then exemplify it by cubics and quartics. Let the given quintic be:

$$x^5 - px^4 + qx^3 - rx^2 + sx - t = 0, \tag{1}$$

and the roots be $x_1=a$, $x_2=b$, $x_3=c$, $x_4=d$, $x_5=e$. Then we may write (1) as a determinant with diagonal terms only, thus:

$$\begin{vmatrix} a-x & 0 & 0 & 0 & 0 \\ 0 & b-x & 0 & 0 & 0 \\ 0 & 0 & c-x & 0 & 0 \\ 0 & 0 & 0 & d-x & 0 \\ 0 & 0 & 0 & 0 & e-x \end{vmatrix} = 0. \tag{2}$$

The determinant (2) is the characteristic equation of the simple diagonal matrix:

$$\phi = \begin{Vmatrix} a & 0 & 0 & 0 & 0 \\ 0 & b & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 \\ 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & e \end{Vmatrix}. \tag{3}$$

Suppose this is transformed into a cyclic matrix ϕ' by using the following unitary matrix τ_5 and its inverse τ_5^{-1} in a unitary transformation:

$$\tau_5 = \frac{1}{\sqrt{5}} \begin{Vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & n & n^2 & n^3 & n^4 \\ 1 & n^2 & n^4 & n & n^3 \\ 1 & n^3 & n & n^4 & n^2 \\ 1 & n^4 & n^3 & n^2 & n \end{Vmatrix}, \qquad \tau_5^{-1} = \frac{1}{\sqrt{5}} \begin{Vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & n^4 & n^3 & n^2 & n \\ 1 & n^3 & n & n^4 & n^2 \\ 1 & n^2 & n^4 & n & n^3 \\ 1 & n & n^2 & n^3 & n^4 \end{Vmatrix}; \tag{4}$$

wherein n is one of the complex fifth roots of unity, *i.e.* $n^5=1$, so that

$$\phi' = \tau_5^{-1}\phi\tau_5 = \begin{Vmatrix} A & E & D & C & B \\ B & A & E & D & C \\ C & B & A & E & D \\ D & C & B & A & E \\ E & D & C & B & A \end{Vmatrix}, \tag{5}$$

wherein

$$\begin{aligned} 5A &= a + b + c + d + e, \\ 5B &= a + n^4b + n^3c + n^2d + ne, \\ 5C &= a + n^3b + nc + n^4d + n^2e, \\ 5D &= a + n^2b + n^4c + nd + n^3e, \\ 5E &= a + nb + n^2c + n^3d + n^4e; \end{aligned} \tag{6}$$

or inversely:

$$\begin{aligned} a &= A + B + C + D + E, \\ b &= A + nB + n^2C + n^3D + n^4E, \\ c &= A + n^2B + n^4C + nD + n^3E, \\ d &= A + n^3B + nC + n^4D + n^2E, \\ e &= A + n^4B + n^3C + n^2D + nE. \end{aligned} \tag{7}$$

The transformation being unitary, the characteristic equation of ϕ', namely

$$\begin{vmatrix} A - x & E & D & C & B \\ B & A - x & E & D & C \\ C & B & A - x & E & D \\ D & C & B & A - x & E \\ E & D & C & B & A - x \end{vmatrix} = 0 \tag{8}$$

will have the same roots a, b, c, d, e, as the original equation (1) or (2). Thus by equating the coefficients of the powers of x in (8) after expansion to the corresponding coefficients of the given equation (1), *i.e.*, p, q, r, s, t, we have changed the problem of finding the roots a, b, c, d, e of (1) into the solution of five simultaneous equations involving A, B, C, D, E as unknowns. If these five equations can be solved algebraically for A, B, C, D, E, then the required roots of the given equation (1) can be evaluated through the relations (7).

3. Application to a cubic. For a cubic, the cyclic matrix is

$$\alpha = \begin{Vmatrix} A & C & B \\ B & A & C \\ C & B & A \end{Vmatrix} \tag{9}$$

and the characteristic equation is

$$(10)\qquad \begin{vmatrix} A-x & C & B \\ B & A-x & C \\ C & B & A-x \end{vmatrix} \equiv -(x^3 - px^2 + qx - r) = 0,$$

wherein the given coefficients p, q, r are to be equated to the three scalar invariants of the matrix (9), namely the sum of the leading diagonals, the sum of the leading diagonal first minors and the determinant of the matrix, *i.e.*,

$$(10a)\qquad p = 3A,$$

$$(10b)\qquad q = 3(A^2 - BC),$$

$$(10c)\qquad r = A^3 + B^3 + C^3 - 3ABC.$$

Although (10c) is still a cubic, it can be solved simultaneously with (10a) and (10b), requiring only a quadratic resolvent. Thus eliminating A from (10b) and (10c) in succession, we find

$$(11a)\qquad -3BC = q - 3A^2 = q - p^2/3 = Q, \text{ say,}$$

$$(11b)\qquad B^3 + C^3 = r - A^3 + 3ABC = r - pq/3 + 2p^3/27 = R, \text{ say.}$$

Expressing B in terms of $1/C$ from (11a) and then substituting it into (11b), the following quadratic in $y = B^3$ (or C^3) is obtained:

$$(12)\qquad y^2 - Ry - Q^3/27 = 0,$$

giving as roots or values of B^3 and C^3 the following:

$$(13a)\qquad B^3 = \frac{R}{2} + \sqrt{\frac{R^2}{4} + \frac{Q^3}{27}},$$

$$(13b)\qquad C^3 = \frac{R}{2} - \sqrt{\frac{R^2}{4} + \frac{Q^3}{27}}.$$

Using any pair of corresponding values for B and C given by the above in terms of p, q, r and noting that $A = p/3$, we get, according to (7), the required roots:

$$(14)\qquad \begin{aligned} x_1 &= a = A + B + C, \\ x_2 &= b = A + \omega B + \omega^2 C, \\ x_3 &= c = A + \omega^2 B + \omega C. \end{aligned}$$

wherein ω and ω^2 are the two complex roots of unity; namely $\omega = -\frac{1}{2} + i\sqrt{3/2}$, $\omega^2 = -\frac{1}{2} - i\sqrt{3/2}$; $i = \sqrt{-1}$.
These will be recognized as Cardan's formulas.

4. Application to a quartic. To apply the above procedure to a quartic, consider the cyclic matrix:

$$\beta = \begin{Vmatrix} A & D & C & B \\ B & A & D & C \\ C & B & A & D \\ D & C & B & A \end{Vmatrix}, \tag{15}$$

whose characteristic equation is:

$$\begin{vmatrix} A-x & D & C & B \\ B & A-x & D & C \\ C & B & A-x & D \\ D & C & B & A-x \end{vmatrix} \equiv -(x^4 - px^3 + qx^2 - rx + s) = 0, \tag{16}$$

with the coefficients p, q, r, s given by the four scalar invariants of β as follows:

$$p = 4A, \tag{17a}$$

$$q = 6A^2 - 2(2BD + C^2), \tag{17b}$$

$$r = 4A^3 - 4A(2BD + C^2) + 4C(B^2 + D^2), \tag{17c}$$

$$\begin{aligned} s = {} & A^4 - 2A^2(2BD + C^2) + 4AC(B^2 + D^2) \\ & - B^4 + C^4 - D^4 + 2B^2D^2 - 4BC^2D. \end{aligned} \tag{17d}$$

Although (17d) is still an equation of the fourth degree, in A, B, C, D, it can be solved simultaneously with the others in (17) by a straightforward algebraic reduction, resulting in a cubic resolvent. Thus from (17a) and (17b) by eliminating A, we find:

$$-2(2BD + C^2) = q - 6A^2 = q - 3p^2/8 = 4Q, \text{ say}, \tag{18a}$$

and substituting this into (17c), we get

$$\begin{aligned} 4C(B^2 + D^2) &= r - 4A^3 + 4A(2BD + C^2) \\ &= r + 8A^3 - 2Aq = r + p^3/8 - pq/2 = 8R, \text{ say}. \end{aligned} \tag{18b}$$

Substituting (18a) and (18b) into (17d), we get

$$-B^4 + C^4 - D^4 + 2B^2D^2 - 4BC^2D = s - 3A^4 + A^2q - Ar = 16S, \text{ say}. \tag{19}$$

Also from (18a),

$$-2BD = 2Q + C^2, \quad \text{or} \quad 4B^2D^2 = (2Q + C^2)^2, \tag{20a}$$

and from (18b),

$$B^2 + D^2 = 2R/C, \quad \text{or} \quad B^4 + 2B^2D^2 + D^4 = (2R/C)^2, \tag{20b}$$

yielding on subtraction:

$$B^4 - 2B^2D^2 + D^4 = (2R/C)^2 - (2Q + C^2)^2. \tag{20c}$$

Substituting (20a) and (20c) into (19), we get:

$$C^4 + 2C^2(2Q + C^2) = 16S + (2R/C)^2 - (2Q + C^2)^2,$$

which, on multiplying by $1/C^2$ and collecting terms, gives

$$4C^6 + 8QC^4 + (4Q^2 - 16S)C^2 - 4R^2 = 0.$$

With $z = C^2$, this becomes a cubic in z, as follows:

$$z^3 + 2Qz^2 + (Q^2 - 4S)z - R^2 = 0. \tag{21}$$

This will be recognized as the Descartes' resolvent for a quartic. Having found C as the principal square root of the resolvent cubic (21), the values of $(B+D)$ and $(B-D)$, and hence B and D, may be evaluated from (20a) and (20b) in terms of A, Q, R, S and then in terms of p, q, r, s. Finally the roots of the quartic (16) are by (7) as follows:

$$\begin{aligned} x_1 &= a = A + B + C + D, \\ x_2 &= b = A + iB - C - iD, \\ x_3 &= c = A - B + C - D, \\ x_4 &= d = A - iB - C + iD. \end{aligned} \tag{22}$$

wherein $i = \sqrt{-1}$ as usual.

5. Concluding remarks. The procedure fails in the case of quintics or equations having a degree higher than the fourth, as it should, since it is not possible to solve such an equation by algebraic methods. The reason for the failure of the present method is that after the unitary transformation and elimination, we are still left with an equation which is of a higher degree than the fifth and which is not amenable to algebraic solutions. However, it would be of interest to study carefully the set of simultaneous equations after having made the unitary transformation to see if some constraining condition or conditions among the coefficients of the given equation might not be found which would be sufficient to permit the present procedure to succeed.

It would hardly be necessary to mention that the method as above given is not limited to real values of the coefficients p, q, r, s, nor to real roots a, b, c, d. Finally, it may be noted that the method also works in case of the quadratic, but the reduction is so simple that the case looks rather trivial.

Lorelei. So flowered the beauteous body of a new geometry [Saccheri's], mermaid-like, the latter portions somewhat fishy, but oh! the elegant torso.—G. B. Halsted (1920).

A difference of opinion. Geometry is not an experimental science.—Poincaré (1899).
We may in fact regard geometry as a branch of physics.—Einstein (1920).
Never in the world were two opinions alike.—Montaigne. —*Contributed.*

DISCUSSIONS AND NOTES

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The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

ON THE SECOND DERIVATIVE

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When the notion of the second derivative is first encountered in the calculus, the examples and exercises are usually of such a character that the second derivative can be simplified by making use of the original function. As far as is known, no general principle has been enunciated for this occurrence.

THEOREM. *If $F(x, y)$ is a homogeneous function of order n in x and y, then $F(x, y)$ is a factor of the second derivative of y with respect to x of the equation $F(x, y) = c$.*

By Euler's theorem on homogeneous functions, we have

$$xF_x(x, y) + yF_y(x, y) = nF(x, y),$$

or more shortly,

$$xF_x + yF_y = nF, \tag{1}$$

where F_x denotes the partial derivative of $F(x, y)$ with respect to x. Differentiating the identity (1) partially with respect to x, and then also with respect to y, we obtain the identities:

$$xF_{xx} + yF_{xy} = (n - 1)F_x, \tag{2}$$

$$xF_{xy} + yF_{yy} = (n - 1)F_y. \tag{3}$$

Elimination of the coefficients x and y in (1), (2), and (3), yields the identity

$$F_x^2F_{yy} - 2F_xF_yF_{xy} + F_y^2F_{xx} = nF(F_{xx}F_{yy} - F_{xy}^2)/(n - 1), \tag{4}$$

assuming $n \neq 1$, and $F_x \neq 0$. (For either of these cases the theorem is evidently true, although trivial.) From the equation $F(x, y) = c$, we have

$$y'' = -\frac{F_x^2F_{yy} - 2F_xF_yF_{xy} + F_y^2F_{xx}}{F_y^3};$$

and using (4), this becomes

$$y'' = -\frac{nF(F_{xx}F_{yy} - F_{xy}^2)}{(n - 1)F_y^3}, \tag{5}$$

as desired.

CHECKING THE SAS CASE IN TRIGONOMETRY

HOWARD EVES, Syracuse University

The favored logarithmic solution of the SAS (two sides and the included angle) case in trigonometry utilizes the law of tangents. If the two given sides are taken as a and b and the given included angle as C, then the formulas successively employed are

$$A + B = 180^\circ - C, \qquad \tan \tfrac{1}{2}(A - B) = [(a - b)/(a + b)] \tan \tfrac{1}{2}(A + B),$$
$$A = \tfrac{1}{2}(A + B) + \tfrac{1}{2}(A - B), \qquad B = \tfrac{1}{2}(A + B) - \tfrac{1}{2}(A - B),$$
$$c = a \sin C/\sin A. \tag{1}$$

Most textbooks of the subject then offer the formula

$$c = b \sin C/\sin B \tag{2}$$

as a check. It is, of course, granted that this formula does not check c itself, but presumably does check A, B, and $\log c$. Therefore, if caution is used in finding c from its logarithm, the whole problem is believed to be thoroughly checked.

It is the purpose of this note to show that it is very easy (and, as a matter of fact, relatively common among trigonometry students) to get incorrect values for A, B, and c and still have the check "check."

To this end suppose that at the very start of the solution a mistake was made in obtaining $A+B$, but that from there on everything was done correctly. Let us designate the parts (now incorrect) thus found by A', B', c'. Then

$$\begin{aligned}
\frac{\sin A'}{\sin B'} &= \frac{\sin [\frac{1}{2}(A' + B') + \frac{1}{2}(A' - B')]}{\sin [\frac{1}{2}(A' + B') - \frac{1}{2}(A' - B')]} \\
&= \frac{\sin \frac{1}{2}(A' + B') \cos \frac{1}{2}(A' - B') + \cos \frac{1}{2}(A' + B') \sin \frac{1}{2}(A' - B')}{\sin \frac{1}{2}(A' + B') \cos \frac{1}{2}(A' - B') - \cos \frac{1}{2}(A' + B') \sin \frac{1}{2}(A' - B')} \\
&= \frac{\tan \frac{1}{2}(A' + B') + \tan \frac{1}{2}(A' - B')}{\tan \frac{1}{2}(A' + B') - \tan \frac{1}{2}(A' - B')} \\
&= \frac{\tan \frac{1}{2}(A' + B') + [(a - b)/(a + b)] \tan \frac{1}{2}(A' + B')}{\tan \frac{1}{2}(A' + B') - [(a - b)/(a + b)] \tan \frac{1}{2}(A' + B')} \\
&= \frac{1 + (a - b)/(a + b)}{1 - (a - b)/(a + b)} \\
&= a/b.
\end{aligned}$$

Therefore, by the law of sines,

$$\sin A'/\sin B' = \sin A/\sin B.$$

A glance at formulas (1) and (2) now shows that the check will "check" for our

incorrect results A', B', and c', for all we are doing is replacing the denominators in (1) and (2) by numbers proportional to them.

As before mentioned, a mistake in finding $A+B$ is common enough among trigonometry students to consider this usual check as unsatisfactory. The remedy, however, is simple. In addition to carrying out the usual check, also check on the sum of the final angles A, B, C. This augmented check, along with care exercised in finding c from $\log c$, may be considered sufficiently good. Of course a longer, but sound, check can be given by using one of the Mollweide relations.

A NOTE ON AREAS

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It is well known (*cf.*, for example, Goursat-Hedrick, *Mathematical Analysis*, v. 1, §94) that if x and y are differentiable functions of a parameter t, the integral

$$I = \frac{1}{2}\int_{t_1}^{t_2} (x\,dy - y\,dx) \tag{1}$$

represents the vectorial area bounded by the curve $x=x(t)$, $y=y(t)$, and the lines OP_1, OP_2, where P_1, P_2 are the points of the curve corresponding to t_1 and t_2. It is a simple matter to transform this integral into the usual expression for area in polar coordinates.

In one of my classes we arrived at the following striking transformation of the integral, which (though I have made no extensive search) I think I have not seen in print.*

If the parameter t is the ratio y/x, the integral (1) *takes the form*

$$I = \frac{1}{2}\int_{t_1}^{t_2} x^2\,dt.$$

For at once, if $y/x=t$, $x\,dy-y\,dx=x^2\,dt$.

As an illustration, take the folium of Descartes, whose parametric equations

$$x = 3at(1+t^3)^{-1}. \qquad y = 3at^2(1+t^3)^{-1}$$

are known to every undergraduate. We have then for the area bounded by the loop of the curve

$$A = \frac{9}{2}a^2\int_0^{\infty} (1+t^3)^{-2}t^2\,dt = \frac{3}{2}a^2.$$

The labor involved here compares favorably with that of the usual process of expressing $\int y\,dx$ in terms of t.

* While this note is in press I learn that the result is given by de la Vallée Poussin, Cours d'analyse infinitesimale, 1938 edition, vol. 1, p. 320, ex. 7. I still think that it deserves to be better known.

The parameter $t=y/x$ is frequently used. It rationalizes any curve whose equation has the form

$$p_n(x, y) - p_{n-1}(x, y) = 0,$$

where p_r is a homogeneous polynomial of degree r; that is, any curve having a singularity of multiplicity $n-1$ at the origin.

Incidentally, will some reader contribute a historical note on the universal acceptance of the symbol

$$\int Pdx + Qdy$$

without parentheses?

NOTE ON THE SOLUTION OF DIOPHANTINE EQUATIONS

NATHAN MANTEL, Wright Field, Ohio

Given an equation of the type $ax-by=c$, where a, b, and c are integers, and a and b have no factors in common, we are to find the least positive integral solutions for x and y. A general procedure has already been developed for this problem. The procedure shown below is felt to be a good deal more efficient than the established procedure. As an example let us take the following equation

$$4397x - 3112y = 2569.$$

We may now state the problem as that of trying to find a multiple of 4397 which leaves a remainder of 2569 when divided by 3112.

We have

$$4397 = 3112 + 1285 \tag{1}$$

$$-4397 = -2(3112) + 1827. \tag{2}$$

Since $1827-1285=542$, equation (2) minus equation (1) gives

$$-2(4397) = -3(3112) + 542. \tag{3}$$

The procedure consists in applying the division algorithm to the remainders to obtain a suitable multiple of 4397 to add to a multiple already found. Thus $1285-2(542)=201$ and equation (1) minus 2 times equation (3) gives

$$5(4397) = 7(3112) + 201. \tag{4}$$

Again $1827-9(201)=18$, and equation (2) minus 9 times equation (4) gives

$$-46(4397) = -65(3112) + 18. \tag{5}$$

Finally $201-11(18)=3$, and equation (4) minus 11 times equation (5) gives

$$511(4397) = 722(3112) + 3. \tag{6}$$

The division algorithm is also applied to 2569 and those of the above remainders found suitable to use. Thus $2569=1285+1284$, $1284=6(201)+78$, $78=4(18)+6$, and $6=2(3)$; or $2569=5(542)-141$, $-141=-8(18)+3$. In the first instance $2569=1285+6(201)+4(18)+2(3)$, and 1, 6, 4, 2 are the proper multipliers of equations (1), (4), (5), (6), respectively, which give us the value of x. Thus $x=1+6(5)+4(-46)+2(511)=869$. In the second instance $2569=5(542)-8(18)+1(3)$ and $x=5(-2)+(-8)(-46)+1(511)$. Had the value of x been negative or greater than 3112, the least positive value of x could have been obtained by adding or subtracting an appropriate multiple of 3112. The value of y can be obtained by substitution or simultaneously with x.

The following is the appearance of the work sheet for this problem:

$$4397x - 3112y = 2569$$

					x		y
4397	0			1	1	1285	1
3112	2569				−1	1827	−2
1285					−2	542	−3
1827		2569		6	5	201	7
542		1285	1 1	4	−46	18	−65
1084		1284		2	511	3	722
201		1206	30 42				
1809		78					
18		72	−184 −260				
198		6					
3		6	1022 1444				
			$x=869$ $y=1227$				

This entire procedure is a matter of about 3 minutes. Experience has shown that a solution for one of the unknowns can be obtained within 15 minutes in cases where a and b are 7 digit integers.

Interesting integers. Professor G. H. Hardy, the English mathematician, had been riding in taxi-cab No. 1729, and remarked to Ramanujan, the gifted Indian, that the number seemed rather a dull one. "No," was the reply, "it is a very interesting number; it is the smallest number expressible as a sum of two cubes in two different ways."

One might conjecture that there is an interesting fact concerning each of the positive integers. Here is a "proof by induction" that such is the case. Certainly 1, which is a factor of each positive integer, qualifies, as do 2, the smallest prime; 3, the smallest odd prime; 4, Bieberbach's number; *etc.* Suppose the set S of positive integers concerning each of which there is no interesting fact is not vacuous, and let k be the smallest member of S. But this is a most interesting fact concerning k! Hence S has no smallest member and therefore is vacuous. Is the proof valid? —*E. F. Bechenbach.*

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

REMARKS ON A VARIATION OF NEWTON'S METHOD

1. Schwerdtfeger's recurring sequences. In a letter to the editor, H. Schwerdtfeger of the University of Adelaide, South Australia, reports a method for obtaining an approximate numerical solution of an analytic equation, which he has used since 1942 in his engineering mathematics classes. He and D. R. Blaskett have reported to the Australian Mathematical Society on the connection between their method and the power series for the inverse of an analytic function in the neighborhood of a root. The method, although superficially different, is essentially similar to that discussed by the editor in *A Variation of Newton's Method.* [This MONTHLY, vol. 51, No. 1 (1944), pp. 36–38.]

To find a simple real root α of an equation $\phi(x)=0$, defined by an analytic function $\phi(x)$ such that $\phi'(\alpha)\neq 0$, Schwerdtfeger rewrites the equation in the form $x=f(x)$, and sets up a recurring sequence of numbers a_n defined by a recursion formula

$$a_{n+1} = f(a_n), \qquad n = 0, 1, 2, \cdots, \alpha = f(\alpha), \tag{1.1}$$

where a_0 is a sufficiently close approximation to the required root α so that

$$|f'(x)| < 1-\epsilon \quad \text{for} \quad |x-\alpha| \leqq |a_0-\alpha|. \tag{1.2}$$

Although this condition insures the convergence of the sequence $\{a_n\}$ to the root α, more rapid convergence is obtained by adding the condition $f'(\alpha)=0$.

To obtain a function $f(x)$ such that $f(\alpha)=\alpha$, $f'(\alpha)=0$, Schwerdtfeger first proves the existence of two functions $A(x)$ and $B(x)$, analytic at $x=\alpha$, such that

$$A(x)\phi(x) - B(x)\phi'(x) = 1. \tag{1.3}$$

He then defines the function

$$f(x) = x + B(x)\phi(x) = x - [\phi(x)/\phi'(x)][1 - A(x)\phi(x)], \tag{1.4}$$

which has the required properties. The graph $y=f(x)$ crosses the line $y=x$ with zero slope at the point $x=\alpha$, so the sequence (1.1) converges rapidly to α. Still more rapid convergence is obtained by placing additional conditions on the function $A(x)$ of (1.3), so that $f''(\alpha)=0$, and perhaps also $f'''(\alpha)=0$, *etc.*

2. Bailey's rational approximation to the *m*th root of a positive number. As a special case of this method, Schwerdtfeger refers to a formula given without proof by V. A. Bailey [*Prodigious calculation*, Australian Journal of Science,

vol. 3, No. 4, 1941, pp. 78–80] for approximating rationally to a root α of the equation

$$\phi(x) \equiv x^m - c = 0, \qquad c = \alpha^m > 0. \tag{2.1}$$

Bailey's function,

$$f(x) = x\frac{(m-1)x^m + (m+1)c}{(m+1)x^m + (m-1)c}, \tag{2.2}$$

is such that

$$f'(x) = (m^2-1)\left[\frac{x^m - c}{m(x^m+c) + (x^m-c)}\right]^2. \tag{2.3}$$

Hence $f(\alpha)=\alpha$, $f'(\alpha)=f''(\alpha)=0$. Successive values a_n obtained by this formula converge rapidly to the required root. For instance, to solve $\phi(x)\equiv x^3-2=0$, take $a_0=1$. Then

$$f(x) = x\left(\frac{2x^3+8}{4x^3+4}\right), \qquad a_1 = f(a_0) = \frac{5}{4}, \qquad a_2 = f(a_1) = \frac{1905}{1512}. \tag{2.4}$$

The approximation a_2 is accurate to six significant figures, whereas a_3 would be accurate to eighteen significant figures.

It is interesting to note that Bailey's function may be obtained by setting $y=x^m-c$ in the approximating formula given in the editor's paper *A Variation of Newton's Method*, namely,

$$f(x) = x + \frac{-y}{y' + (y''/2)(-y/y')}. \tag{2.5}$$

3. Estimates of the error. Let us denote the relative error in the nth approximation to α by E_n, so that

$$E_n = \frac{a_n}{\alpha} - 1, \qquad E_{n+1} = \frac{f(a_n)}{\alpha} - 1. \tag{3.1}$$

Then if in (2.5) we expand the function y in a Taylor series about the point $x=\alpha$, we have

$$y = c_1(x-\alpha) + c_2(x-\alpha)^2 + c_3(x-\alpha)^3 + \cdots, \tag{3.2}$$

$$\frac{f(x)}{\alpha} - 1 = \left(\frac{x}{\alpha} - 1\right)^3\left(\frac{\alpha}{c_1}\right)^2(c_2^2 - c_1c_3) + \text{higher order terms.} \tag{3.3}$$

Hence

$$E_{n+1} = E_n^3\left(\frac{\alpha}{c_1}\right)^2(c_2^2 - c_1c_3) + \text{higher order terms in } E_n. \tag{3.4}$$

This formula gives a good estimate of the relative errors in the successive approximations, and shows that with each new approximation the number of known correct significant figures in the root α is about tripled. In the special case of Bailey's function (2.2) the estimate of relative error is

$$E_{n+1} = E_n^3(m^2 - 1)/12 + \text{higher order terms in } E_n. \tag{3.5}$$

4. Calculation of roots by machine. Bailey's function (2.2) should prove extremely valuable for the accurate determination of mth roots by a calculating machine. From a table of mth powers an approximation x^m to the given number c is found, such that x approximates $\sqrt[m]{c}$ to three significant figures. A single division on the machine then suffices to determine the root to nine or ten significant figures. The estimate (3.5), computed by slide rule or by machine to two or three significant figures, serves to indicate more precisely the magnitude of the error of this second approximation.

CLUB REPORTS 1944–45

Advisers and officers of Mathematics Clubs are requested to send in their annual reports for 1944–45 to the editor as soon after the school year as possible. Reports should include the titles of talks followed by the names of the speakers, an account of noteworthy events or activities or prizes awarded, and should conclude with the names of the officers.

CLUB REPORT 1943–44

Mathematics Society, Brooklyn College

The Mathematics Society of Brooklyn College completed the Spring term of its 1944 program with pride in its fifty active members who worked on a term topic under the guidance of Dr. Kormes, the faculty adviser. The topic was *Fourier series and orthogonal functions*, with emphasis on the modern treatment based upon the concept of function-space. The following lectures were delivered:

General introduction, by Dr. Kormes
Examples of Fourier expansions, by S. Greif
Orthogonal functions, by J. Greenstadt
Fourier's theorem, by G. Washnitzer
Physical applications, by W. Hauser
Physical applications, by N. Potter
Orthogonal spaces, by S. Shecter.

Since many of the members will be in the armed services shortly, Dr. Kormes suggested that a committee be organized to communicate with these members so that the spirit of the Society may be continued. In this way the members in the armed forces will be kept in closer contact with their mathematics. The suggestion was immediately adopted by the Society.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Elementary Statistics with General Applications. By M. M. Blair. New York, Henry Holt and Company, Inc., 1944. 14+690 pages. $3.50.

This book is designed to present a large body of statistical theory and practice in a practical and non-technical way. The author "has chosen and limited his materials for the purpose of meeting the needs of that large and ever-increasing body of students and businessmen who must have an elementary knowledge of statistics. . . . No effort is made to introduce the student to the derivation of formulas." The plan of the book seems to be to present complete worksheets for the solution of those problems usually included in elementary statistics. The examples chosen are chiefly from the fields of agriculture, business, education, and sociology.

The book is divided into five parts: 1. General Methods.—2. Large Sample Analysis.—3. Time Series Analysis.—4. Small Sample Analysis.—5. Curvilinear and Multiple Analysis.

It is deplorable that the manuscript was not edited before publication by a competent mathematician, if not by a mathematical statistician. The chapters are replete with misuse of mathematical notation and terminology, as well as actual errors. To illustrate, equal signs are frequently used as punctuation marks; a "power" is a positive integral exponent; an "exponent" is a positive fractional exponent; a "ratio" is a proportion, and a "rate" is a "mixed ratio"; any curve is a "line"; any integer is an "even" number; *etc.*

Some of the errors might be mentioned. The least squares line (page 243) is not necessarily "a line which on the average comes nearest to all the points or items of data." In Figure 53, the intercept a is drawn incorrectly. The denominator in Formula 72 should be $\sum p_0 q_0$. One can not say (page 481) that "the chances are 20 to 1 that the two small samples were drawn from separate parent populations" simply because the null hypothesis gives those odds against obtaining the given difference. The calculation of S_y (page 554) assumes $\sum Z = 0$, contrary to fact. The signs of a and b in the parabola $Y = a + bX + cX^2$ (page 556) depend on the position of the origin, not on the shape of the curve. A similar statement is true for the cubic parabola (page 565). Since the sum of the deviations from the arithmetic mean is zero, all the summations on page 582 vanish.

The book is further marred by many arithmetical errors in the examples, as well as by a large number of typographical errors. In the opinion of the reviewer, this text can not be recommended in its present form.

H. D. LARSEN

Methods of Advanced Calculus. By Philip Franklin. New York and London, McGraw-Hill Book Company, Inc., 1944. 12+486 pages. $4.50.

As is well-known, this text-book is the second written by Professor Franklin on the subject of Advanced Calculus, the first having been published in 1940 by John Wiley and Sons under the title of "A Treatise on Advanced Calculus." The two books are in some respects different as to content, but in all respects different as to point of view.

The reader, familiar with the first book, will recall that it was in the nature of an introduction to the theory of functions of a real and complex variable. As such, it began with a discussion of the real number system, the Bolzano-Weierstrass and Heine-Borel theorems, and limits and continuity, before considering the more usual topics of an advanced calculus course.

While the first book was thus a sort of "Cours d'Analyse," the new book, to quote from its introduction, "has two principal objectives: first, to refresh and improve the reader's technique in applying elementary calculus; second, to present those methods of advanced calculus which are most needed in applied mathematics." Thus, the new book has the purpose of serving the needs of prospective engineers, physicists, and others who may regard mathematics primarily as a tool. Accordingly, it attempts to include as many topics of possible practical interest as could be crowded into a space of somewhat less than five hundred 5×8″ pages and treat these topics with special stress on the manipulative and numerical aspects.

The book opens with two chapters that include a definition of the elementary functions of a complex variable, a treatment of partial differentiation and implicit functions, a statement of Taylor's series for functions of one complex variable and of two real variables, and incidentally a discussion of Newton's method for the approximate solution of equations, as well as a discussion of determinants and linear equations (but not of matrices). The third chapter begins with a study of vector analysis, with applications to the theory of space curves and surfaces—a study which is completed in the eighth chapter with a treatment of the divergence theorem and Stokes', Green's and Gauss' theorems. Chapters IV to VII are concerned with formal and approximate integration, multiple integrals, the Gamma function and other definite integrals, elliptic integrals, and line integrals. In connection with the last topic, there is developed a bit of the theory of analytic functions of a complex variable—the Cauchy-Riemann differential equations, the Cauchy integral and the residue theorem with applications to the evaluation of definite integrals, and conformal mapping by means of the Schwarz transformation. Chapters IX to XI deal with ordinary and partial differential equations and boundary value problems involving expansions in terms of Legendre polynomials, Bessel's functions, and Fourier series. Finally, Chapter XII provides an introduction to the calculus of variations and Lagrange's equations.

Like the "Treatise," the new book gives general references and a list of about

100 exercises at the close of each chapter. Some of these exercises are simple applications of the text, but others have the admirable purpose of providing an opportunity for the student himself to develop some non-trivial material not given explicitly in the text. Answers are given to all the exercises—a feature which may be helpful to engineers and others attempting to read the book without the aid of an instructor.

In order to include so many topics, the new book intentionally gives some intuitive or heuristic "proofs" and omits others altogether. To save space, it occasionally also omits minor and sometimes important details. For example, on page 3, after defining e^z, sin z and cos z for complex z by means of their Maclaurin expansions, the author says: "Convergent series of this type may be multiplied and added together in the same way that polynomials are combined. It follows that the functions defined by the series satisfy the relations $e^{z_1+z_2}=e^{z_1}e^{z_2}$ as well as the addition theorems sin $(z_1+z_2)=\cdots$." Also, on page 4, the book fails to give any definitions of limit and derivative of a function of a complex variable prior to the tabulation of the answers for the derivatives of elementary functions of a complex variable. Of course, these details can be supplied in class by the instructor, their omission causing a hardship principally to a student who wishes to read the book on his own.

The statements of the theorems in the book seem, on the whole, to be accurate. An exception is the following made on page 27 regarding the approximation r_1 to a root r of an algebraic equation $f(x)=0$. "Usually, if r_1 is correct to n decimals, $f(r_1)$ will be less than $1/10^n$." An example contradicting this statement is the equation $f(x)=400-600x+200x^2=0$, which has the exact root $r=2$, but, for the approximation $r_1=2.001$, we find $f(r_1)=0.6$.

Indeed, it is not the accuracy but the manner in which some of the theorems are stated that may be open to objection. For example, in chapter VIII, the details of the theorems are dispersed in the main body of the text, instead of being printed compactly in some distinctive type. While stating theorems in the more traditional manner would have detracted from the informality of the presentation, it would have enhanced the reference value of the text. This would be especially true if, in addition to being stated in italics, each unproved theorem had been accompanied by a specific page reference to some other book containing a proof.

Despite this objection, the new volume may be regarded as one of the best text-books now available for any advanced calculus course which is intended to be a terminal course in mathematics for engineers, physicists and the like. However, for prospective mathematicians and others for whom the advanced calculus is but a stepping stone to higher mathematics, there is no question that Franklin's first book, the "Treatise," still is a better choice.

MORRIS MARDEN

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 666. *Proposed by N. D. Lane, St. Andrew's College, Aurora, Ont.*

For what value of k is the curve $x^m y^n = k$ self-reciprocal with respect to the circle $x^2+y^2=1$?

E 667. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let P, Q, R, P', Q', R' be arbitrary points on the respective edges BC, CA, AB, DA, DB, DC of a given tetrahedron $ABCD$. Prove that four planes, parallel to the faces BCD, CDA, DAB, ABC, drawn through the centroids of the respective tetrahedra $AQRP'$, $BRPQ'$, $CPQR'$, $DP'Q'R'$, form a tetrahedron of constant volume.

E 668. *Proposed by Walter Penney, Navy Department, Washington, D. C.*

Prove that the equation $x^2-3y^2=17$ has no solution in integers.

E 669. *Proposed by J. H. Butchart, Grinnell College*

Let G be the centroid of n coplanar points P_i, Q any point in the same plane, and Δ_i the signed area of the triangle QGP_i. Show that

$$\sum \Delta_i = 0.$$

E 670. *Proposed by C. D. Olds, Purdue University*

Sum the series

$$\sum_{r=1}^{2n-1} (-1)^{r-1} r \Big/ \binom{2n}{r}.$$

SOLUTIONS

Curves of Constant Width

E 610 (*b*) [1944, 88 and 532]. *Proposed by Howard Eves, Syracuse University*

What is the least area that a closed curve of constant diameter d may have?

Remarks by R. P. Boas, Jr., Harvard University, and Henry Scheffé, Syracuse

University. The least area is $\frac{1}{2}(\pi-\sqrt{3})d$, attained only by the "Reuleaux triangle" formed by three arcs of circles of radius d with centers at the vertices of an equilateral triangle of side d. For a proof, and references to earlier proofs, see A. E. Mayer, *Der Inhalt der Gleichdicke*, Math. Annalen, vol. 110 (1934), pp. 97–127 (especially p. 115), or Bonnesen and Fenchel, *Theorie der konvexen Körper*, Berlin, 1934 (Ergebnisse series, no. 3). Nearby portions of the latter book discuss many interesting properties of curves of constant width and some of their generalizations (*e.g.*, solids of constant width).

Two Fours

E 631 [1944, 405]. *Proposed by J. A. Tierney, Norwich University, Northfield, Vt.*

Express the number 64, using two fours (and the operations of arithmetic).

I. *Solution by 4 members of the Mathematics Dept. of the Manhattan High School of Aviation Trades.*

$$64 = \sqrt{\sqrt{\sqrt{4^{4!}}}} = \sqrt{4^{\Gamma(4)}} = 4^{[\log 4!]} = \{\phi(4!)\}^{\sqrt{4}} = 4^{\sqrt{\pi(4!)}}$$

(where $\pi(x)$ is the number of primes not exceeding x), or 44 in the quindenary scale.

II. *Solution by D. H. Browne, Buffalo, N. Y.* $\phi(\phi(4^4))$.

III. *Solution by Howard Eves, Syracuse University.* $4^{\sqrt{4!}}$.

Also solved by W. E. Buker, L. H. Bunyan, L. R. Chase, A. L. Lanckton, H. D. Larsen, E. D. Schell, Sol Rubinon, Jeanette Van Os, Alan Wayne, R. H. Wilson, Jr., F. E. Wood, and Thomas Wyman.

An unpublished manuscript of the late W. W. Rouse Ball contains an expression for every positive integer up to 90 in terms of three fours. Two fours suffice in many cases.

Coaxal Spheres

E 633 [1944, 405]. *Proposed by N. A. Court, University of Oklahoma*

Given a point M and four spheres (A), (B), (C), (D) whose centers form a tetrahedron, let MEE' be the transversal from M to the two opposite edges BC, DA, and let spheres (E) and (E') be constructed coaxal with the pairs of spheres (B), (C), and (D), (A). We have analogous spheres (F) and (F'), (G) and (G'). Show that the sphere (M) coaxal with (E), (E') is likewise coaxal with (F), (F') and with (G), (G').

Solution by the Proposer. The sphere (E), being coaxal with the spheres (B), (C), is orthogonal to any sphere orthogonal to (B) and (C), and in particular to the orthogonal sphere (R) of the four given spheres (A), (B), (C), (D). The spheres (E'), (F), (F'), (G), (G') are orthogonal to (R) for similar reasons. The sphere (R), being orthogonal to (E), (E'), is also orthogonal to the sphere (M) coaxal with these spheres. The three spheres (F), (F'), (M) are orthogonal to the same sphere (R), and their three centers are collinear; hence these spheres are coaxal. Similarly for the spheres (G), (G'), (M).

Also solved by Howard Eves.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4154. *Proposed by H. F. Sandham, Trinity College, Dublin*

Find the envelope of the axes of conics inscribed in a quadrilateral.

4155. *Proposed by G. W. Wishard, Norwood, Ohio*

Find a formula for the sum of the series

$$1 + 2^2 + 3^3 + \cdots + n^n.$$

If such a formula is impossible find superior and inferior limits for the sum.

4156. *Proposed by Charles M. Stein, New York*

Show that if $f(u)$ is a polynomial in u and $D=d/dx$, then for any positive integer n

$$\left\{1 + \frac{[(k-1)n]!}{[kn]!}(y^n - x^n)D^n\right\} \cdots \left\{1 + \frac{1}{n!}(y^n - x^n)D^n\right\} f(x^n) = f(y^n).$$

If possible, generalize the above result to analytic functions $f(u)$ as in the special case $n=1$ in 3985 [1942, 345].

4157. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Find the base such that a number of eight digits of the form *ababcdcd* can be the square of a number of four digits *mnmn*, where the numbers of two digits *ab* and *cd*, or *ab* and *mn*, or *cd* and *mn* are consecutive.

SOLUTIONS

Geodesic Curvature

4105 [1944, 49]. *Proposed by J. H. Butchart, Grinnell College*

Let ρ, ρ_s, R be respectively the radius of curvature of a curve, the radius of curvature of the locus of the center of the osculating sphere, and the radius of this sphere. Then the radius of geodesic curvature ρ_g of the locus of the center of curvature with regard to the polar developable is given by $\rho_g = R^3/(2R^2 - \rho\rho_s)$.

Solution by C. E. Springer, University of Oklahoma. If the curve is represented by $x^i=x^i(s)$, $(i=1, 2, 3)$, then $R^2=\rho^2+\tau^2\rho'^2$, where τ is the radius of torsion of the curve, and the prime indicates differentiation with respect to the arc length s. Let α^i, β^i, γ^i be the direction cosines of the tangent, principal normal, and binormal to the curve. Then the radius of curvature ρ_s of the curve generated by the point $x^i+\rho\beta^i-\rho'\tau\gamma^i$ is readily found to be given by

$$\rho_s = \rho + \tau(\rho'\tau)'.$$

The element of arc on the polar developable

$$X^i = x^i + \rho\beta^i + t\gamma$$

is given by $d\sigma^2=Edt^2+2Fdtds+Gds^2$, where

$$E = 1, \qquad F = -\rho/\tau, \qquad G = \frac{\rho^2}{\tau^2} + \left(\frac{t}{\tau} + \rho'\right)^2.$$

From the formula

$$\frac{1}{\rho_g} = -\frac{1}{\sqrt{EG-F^2}}\left(\frac{\partial}{\partial s}\frac{F}{\sqrt{G}} - \frac{\partial}{\partial t}\sqrt{G}\right),$$

(Eisenhart, *Differential Geometry*, 1909, p. 136), the expression for ρ_g with $t=0$ can be calculated. By means of the expressions for R and ρ_s given above, the formula for ρ_g reduces to $R^3/(2R^2-\rho\rho_s)$.

Solved also by P. D. Thomas.

Editorial Note. The solution by Thomas is based on theorems in Eisenhart's *Differential Geometry*, pp. 14, 37, 149; and in Bell's *Coordinate Geometry of Three Dimensions*, pp. 301, 370, 371. The proposer did not give his proof but stated that it is based on developing the polar developable into a plane, the given curve reducing to a point.

The computations in the two solutions are quite lengthy and it appears simpler to use vectors. If P is a point on the given curve with the vector $\mathbf{r}$; $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$ are unit vectors for the tangent, principal normal, and binormal; s is the arc length and prime marks denote derivatives with respect to s; then, if $\mathbf{c}$ denotes the vector of C the center of circular curvature, and $\mathbf{S}$ the vector of the center S of spherical curvature, we have

$$\mathbf{c} = \mathbf{r} + \rho\mathbf{n}, \qquad \mathbf{c}' = \rho\tau\mathbf{b} + \rho'\mathbf{n},$$
$$\mathbf{S} = \mathbf{c} + \sigma\rho'\mathbf{b}, \qquad \rho_S = \rho + \sigma(\sigma\rho')', \qquad R^2 = \rho^2 + (\sigma\rho')^2, \tag{1}$$

where $\kappa=1/\rho$, $\tau=1/\sigma$ are respectively the curvature and torsion at P. Then, if the subscripts denote the corresponding quantities for the locus of C,

$$s_1' = ds_1/ds = \tau R, \qquad R\mathbf{t}_1 = \rho\mathbf{b} + \sigma\rho'\mathbf{n}. \tag{2}$$

If ψ denotes the angle from the positive direction of $\mathbf{n}$ to $\mathbf{t}_1$, then

$$\mathbf{t}_1 = \sin\psi\mathbf{b} + \cos\psi\mathbf{n}, \qquad \psi' = (\rho' R - \rho R')/R\sigma\rho',$$

$$\psi' + \tau = (2R^2 - \rho\rho_S)/R^2\sigma, \tag{3}$$

$$\kappa_1\mathbf{n}_1 = -\frac{\kappa\cos\psi}{\tau R}\mathbf{t} + \frac{\psi' + \tau}{\tau R}[\cos\psi\mathbf{b} - \sin\psi\mathbf{n}].$$

The locus of C lies on the polar developable surface with the normal $\mathbf{t}$ at C, and hence the first term above is the component of the vector curvature $\kappa_1\mathbf{n}_1$ along the normal to the surface, the remaining part is therefore the curvature of the locus of C tangential to the surface, that is the vector geodesic curvature at C. The coefficient of $(\psi'+\tau)/\tau R$ is a unit vector in the direction CS_b, where S_b is the projection of S on the binormal at P and CS_b has the length R. Hence the geodesic curvature of the C locus is

$$\kappa_g = (2R^2 - \rho\rho_S)/R^3.$$

Thus, if $2R^2 > \rho\rho_s$, the locus of C turns from the geodesic tangent at C toward the region in which S_b lies.

A Periodic Product

4107 [1944, 50]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let $N = 123\cdots(n-3)(n-2)n$ be a number of $n-1$ digits in order of increasing magnitude in the system of base $n+1$. The product $P = N\cdot L$ is formed where $L = \alpha\beta$ consists of two digits whose sum $\alpha+\beta=\gamma$ is less than n and δ is the greatest common divisor of n and γ. Show that

$$P = N\cdot L = ab\cdots pqab\cdots pq\cdots ab\cdots pq,$$

where the δ periods $ab\cdots pq$ are formed by n/δ distinct digits.

Dedicated to E. P. Starke.

Solution by E. P. Starke, Rutgers University. (1) $n\cdot N$ is the n-digit number $11\cdots 1$, and $L\cdot n\cdot N$ is the $(n+1)$-digit number $M = \alpha\gamma\gamma\gamma\cdots\gamma\beta$. Hence the desired number $P = N\cdot L = M/n$. From the form of $M = \alpha\gamma\cdots$ it is evident that the first digit a of $P = M/n$ is α if $\alpha+\gamma<n$, but is $\alpha+1$ if $\alpha+\gamma\geqq n$. From the form of N and L, the last digit q of $P = N\cdot L$ must be $n+1-\beta$ unless $\beta = 0$.

(2) Any number is congruent, mod n, to the sum of its digits.

(3) If M is divided by n, the remainders r_i and r_j $(i<j<n)$ of the ith and jth partial divisions are alike if and only if the $(j-i+1)$-digit number $r_i\gamma\gamma\gamma\cdots\gamma(\gamma-r_j)$* is divisible by n. By (2) it is congruent mod n to $\gamma(j-i)$ which is divisible by n if and only if $(j-i)$ is divisible by n/δ. Hence the number M/n consists of δ periods, $a\cdots q$, of n/δ digits each, a and q being given in (1).

(4) For the divisions of the 2-digit numbers $\theta\gamma$, $\theta = 0, 1, \cdots, n-1$, by n, different remainders imply different quotients. Thus the digits within any one period of M/n are distinct. This completes the proof.

The connection of this problem with 3851 [1940, 56] and 4100 [1945, 163] should be noted. The methods of solution are entirely analogous.

* If $r_i > \gamma$, the last two digits are $(\gamma-1)(\gamma+n+1-r_i)$.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Professor T. E. Mergendahl of Tufts College has been elected president of the Association of Teachers of Mathematics in New England.

Professor H. C. Carver and Assistant Professor Samuel Eilenberg of the University of Michigan have been granted leaves of absence, the former for work with the Army Air Forces and the latter to serve with the applied mathematics group at Columbia University.

Associate Professor Will Feller of Brown University has been appointed to a professorship at Cornell University.

Dr. N. A. Hall of Chance Vought Aircraft has been appointed mechanical engineer in charge of thermodynamic analysis for the research division of the United Aircraft Corporation in East Hartford, Connecticut.

Professor W. H. Hill of Kansas State Teachers College of Pittsburg, Kansas, has retired.

Assistant Professor J. E. Ikenberry of Franklin and Marshall College has been appointed professor of mathematics and head of the department at Madison College in Harrisonburg, Virginia.

Professor H. K. Justice of the University of Cincinnati has been granted a leave of absence to serve with the United States Government.

Assistant Professor Wilfred Kaplan of the University of Michigan, Assistant Professor Charles Loewner of the University of Louisville, and Associate Professor S. E. Warschawski of Washington University have been granted leaves of absence to serve with the applied mathematics group at Brown University.

Associate Professor G. E. Moore of the University of Illinois has been appointed assistant dean of liberal arts and sciences.

Associate Professor W. R. Murray of Franklin and Marshall College has been appointed chairman of the department.

Dr. C. D. Olds of Purdue University has been appointed to an assistant professorship at San Jose State College, San Jose, California.

Assistant Professor C. K. Payne of Washington Square College, New York University, has been promoted to an associate professorship.

Dr. L. M. Rauch has been appointed to an assistant professorship at Wesleyan University.

Dr. R. D. Specht of the University of Florida has been appointed to an assistant professorship at the University of Wisconsin.

Dr. R. E. Street of Dartmouth College has been appointed engineer in the General Electric Company at Schenectady.

The following appointments to instructorships have been announced:
Polytechnic Institute of Brooklyn: M. J. Forray
Southeastern Louisiana College (Hammond, Louisiana): Dr. Margaret R. Davis
University of Chicago (College): S. P. Hughart
University of Kentucky: Mrs. Betty D. Crawley
University of Nebraska: Dr. J. F. Heyda

Dean J. L. Gibson of the University of Utah died February 10, 1945. He was a charter member of the Association.

Professor Emeritus W. F. Long of Franklin and Marshall College died January 1, 1945.

M. M. S. Moriarty, formerly a teacher in Holyoke High School, Holyoke, Massachusetts, died April 30, 1944. He was a charter member of the Association.

Professor C. H. Rowe of the University of Dublin died in November 1944.

Professor Emeritus G. T. Sellew of Knox College died December 27, 1944. He was a charter member of the Association.

Dr. J. A. Swenson of Teachers College, Columbia University, died May 2, 1944.

Associate Professor A. L. Underhill of the University of Minnesota died January 18, 1945. He was a charter member of the Association.

P. H. Underwood, formerly a teacher in Ball High School, Galveston, Texas, died December 14, 1944. He was a charter member of the Association.

Professor N. R. Wilson of the University of Manitoba died December 27, 1944.

SUMMER COURSES

The following institutions announce courses in mathematics for the summer of 1945:

Brown University: From June 11 to September 1 the program of advanced instruction and research in mechanics will be in its fifth year. Since this program is supported by outside funds, because of its importance in building up mathematics in the field of applications to engineering, no tuition fees will be charged. The faculty will consist of Professors Stefan Bergman, Lipman Bers, G. E. Hay, Charles Loewner, Willy Prager, S. E. Warschawski, and two others still

to be chosen. A special feature outside the core of material in mechanics is a course in thermodynamics. In addition, the following courses are planned: introduction to partial differential equations, geometrical foundations of mechanics, theory of flight (engineering aerodynamics), elasticity, differential and integral equations of mathematical physics, advanced aerodynamics, advanced theory of vibrations, and special seminars and research in various phases in mechanics.

The Catholic University of America: From July 2 to August 9 the following advanced courses will be offered: By Professor Finan; theory of equations; theory of numbers. By Professor Ramler: advanced euclidean geometry; differential equations; analytic projective geometry. By Professor Rice: advanced calculus; mathematical statistics.

Columbia University: From July 2 to August 10 the following graduate courses will be offered: By Professor Kasner: introduction to modern mathematics; transformations and groups. By Professor Lorch: theory of numbers. By Professor Murray: differential equations. By Professor Ritt: theory of functions of a real variable.

Northwestern University: From June 23 to August 25 the following advanced courses will be offered: higher analysis; theory of statistics; introduction to the theory of numbers; geometry for teachers; the history and teaching of mathematics; theory of differential equations; seminar in analysis. These courses may be taken for six or nine weeks.

The Ohio State University: From June 19 to August 31 the following advanced courses will be offered: By Professor Bamforth: advanced euclidian geometry. By Professor Blumberg: introduction to modern mathematics; introduction to the theory of relativity. By Professor Jones: differential equations. By Professor Mickle: advanced calculus. By Professor Rado: theory of fields.

The State University of Iowa: From June 13 to August 8 the following graduate courses will be offered: By Professor Conkwright: theory of numbers. By Professor Chittenden: differential equations. By Professor Knowler: elements of statistics. By Professor Price: teaching of mathematics. By Professor Wylie: astronomy.

The University of Chicago: The first term of the summer quarter is from June 25 to August 3 and the second term from August 4 to September 15. The following advanced courses will be offered: By Professor Barnard: vector analysis. By Professor Hartung: two courses in the teaching of mathematics. By Professor Hestenes: calculus of variations. By Professor Graves: theory of functions of real variables. By Professor Logsdon: higher plane curves. By Professor Schilling: theory of numbers. These courses will meet four times per week in the first term and the first half of the second term.

The University of Colorado: From July 2 to August 22 a course in power series will be offered in addition to the usual elementary courses. Other advanced courses will be offered if there is sufficient demand.

The University of Michigan: From July 2 to August 24 the following courses will be offered in addition to the standard courses in differential equations, theory of equations and determinants, and advanced calculus: By Professor Anning: modern geometry. By Professor Craig: theory of statistics (second course). By Professor Coe: theoretical mechanics. By Professor Dwyer: theory of statistics (first course) and computational methods. By Professor Fischer: theory of probability and introduction to mathematics of life insurance. By Professor Hildebrandt: theory of functions of a real variable and partial differential equations. By Professor Karpinski: teaching of geometry and history of algebra. By Professor Rainich: algebraic theory and differential geometry. By Professor Wilder: introduction to the foundations of mathematics and combinatorial topology.

The University of Minnesota: From June 20 to July 28 the following advanced courses will be offered: By Professor Gibbens: tutorial course in advanced mathematics. By Professor Koehler: calculus of finite differences. By Professor Olmstead: limits and series. From July 30 to September 1 the following will be given: By Professor Carlson: tutorial course in advanced mathematics. By Professor Koehler: vector analysis.

The University of Pennsylvania: From July 5 to August 30 the following advanced courses will be offered: By Professor Caris: differential equations. By Professor Rademacher: partial differential equations. By Professor Whaples: vector analysis.

The University of Virginia: From July 2 to August 25 the following advanced courses will be offered: By Professor Hedlund: advanced calculus and applied mathematics. By Professor Whyburn: functions of real variables.

The University of Wisconsin: From May 28 to September 15, in addition to the usual elementary courses, the following advanced courses will be offered: By Professor Bruck: projective geometry. By Professor Everett: theory of equations. By Professor Langer: statics, differential equations. From June 23 to August 17 the following advanced courses will be offered: By Professor Evans: mathematics of educational statistics. By Professor Everett: theory of equations. By Professor Langer: mathematical applications. By Professor Trump: college geometry.

The University of Wyoming: From June 20 to July 24 the following advanced courses will be offered: By Professor Barr: differential equations, college geometry. By Professor Neubauer: history of mathematics. From July 25 to August 28: By Professor Barr: differential equations, advanced college algebra. By Professor Neubauer: fundamental concepts.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

NEW LIST OF ESSENTIAL ACTIVITIES

On January 15, 1945, the Selective Service System published a revised list of essential activities. In the introduction to the new Memorandum it is specified that "All technical, scientific and research personnel engaged in any of the activities in the list, whether or not the activity appears in bold type (indicating a critical activity) or regular type, are regarded as being engaged in critical activities." Quotations are given below from items 33 and 35 contained in the list.

33. *Educational services.*—Public and private industrial and agricultural vocational training; elementary, secondary, and preparatory schools; junior colleges, colleges, universities, and professional schools, educational and scientific research agencies; United States Maritime Service training program; Civil Aeronautics Administration Civilian Pilot Training Program; armed forces contract flying, ground and factory aviation schools; and the production of technical and vocational training films.

35. *Technical, scientific, and management services.* The supplying of technical, scientific and management services to establishments engaged in war production; union-management negotiation services; publication of technical and scientific books and journals.

FEDERATION OF WOMEN'S CLUBS METRIC RESOLUTION

At the annual convention of the General Federation of Women's Clubs held April 25–28, 1944, in St. Louis, the following resolution was introduced and adopted unanimously by the delegates. This organization represents 16,500 clubs and 2,500,000 individual members.

Whereas, the irregular, numerous, unwieldy, and complicated units of weights and measures used in the United States and Great Britain are a hindrance to the teaching of arithmetic, every day commercial transactions, and world trade, and

Whereas, the metric system of weights and measures has only three units; meter, liter, and gram, interrelated and decimally divided like our dollar, and

Whereas, the metric system is now used in the United States in science, some factories, jewelry and optical industries, all electrical and radio measurements, athletic events, some hospitals and government departments, and especially at present in the manufacture of ammunition, and

Whereas, the Council on Pharmacy and Chemistry of the American Medical Association has recently decided that henceforth it will use only the metric system, and

Whereas, the gradual introduction of the metric system in this country (exactly as it has been introduced in 55 other countries) is feasible, and

Whereas, the full adoption of the metric system by the United States would be of great benefit to this country in post-war reconstruction, in promoting international commercial relations, particularly with the countries of Latin America, Continental Europe and Asia, therefore be it

Resolved, that the General Federation of Women's Clubs in Convention assembled, April, 1944, endorses legislation in Congress for the nation-wide adoption of the metric system of weights and measures.

The foregoing resolution was drawn up and presented by Eleonore F. Hahn (Mrs. Otto Hahn), member of the Board of Directors of the General Federation of Women's Clubs.

NOTES ON THE NAVY V-12 PROGRAM

On February 17, 1945, President Roosevelt signed a bill authorizing an increase in the size of the Navy ROTC. At present there are twenty-seven NROTC organizations with an enrollment of approximately 6,500 men. The new measure increases the limitation on the size of the NROTC from 7,200 to 24,000 until one year after the cessation of hostilities; thereafter the upper limit will be 14,000. The number and location of the new ROTC units will be determined by a special committee of educators and Naval officers. It is expected that the additional units will be in operation by July 1, 1945.

The Navy also plans to increase the standards of the NROTC program to the level prevailing before the war. This means that the curriculum will be placed on an eight-term basis, thereby enabling students to complete their college education before being commissioned.

As a consequence of this reorganization of the Navy College Training Program, it is expected that a large proportion of the V-12 personnel will be transferred to the NROTC by July 1. Some special categories, however, will probably continue under the V-12 Program. Among these will be Marine Corps trainees, medical and theological students, physics majors, and aerology specialists.

THE MATHEMATICAL TABLES PROJECT

As a supplement to the article "The Mathematical Tables Project," which appeared in this MONTHLY for November, 1944, a complete list follows of all articles containing short tables prepared under the supervision of members of the Mathematical Tables Project up to January 1, 1945.

1. On the computation of the second differences of the $Si\ (x)$, $Ei\ (x)$, and $Ci\ (x)$ functions, by A. N. Lowan, Bull. Amer. Math. Soc., vol. 45 (1939), No. 8.

2. Errors in Hayashi's table of Bessel functions for complex arguments, by A. N. Lowan and Gertrude Blanch, Bull. Amer. Math. Soc., vol. 47 (1941), No. 4.

3. The internal temperature-density distribution of the sun, by G. Blanch, A. N. Lowan, R. E. Marshak, and H. A. Bethe, Astrophys. J., vol. 94 (1941), No. 1.

4. Table of the zeros of the Legendre polynomials of order 1–16 and the weight coefficients for Gauss' mechanical quadrature formula, by A. N. Lowan, Norman Davids, and Arthur Levenson, Bull. Amer. Math. Soc., vol. 48 (1942), No. 10.

5. A table of coefficients for numerical differentiation, by A. N. Lowan, H. E. Salzer, and A. Hillman, Bull. Amer. Math. Soc., vol. 48 (1942), No. 12.

6. On the function $H(m, a, x) = \exp(-ix)\ F(m+1-ia,\ 2m+2;\ ix)$, by A. N. Lowan and William Horenstein, J. Math. and Phys., vol. 21 (1942), No. 4.

7. Table of integrals $\int_0^x J_0(t)dt$ and $\int_0^x Y_0(t)dt$, by A. N. Lowan and Milton Abramowitz, J. Math. and Phys., vol. 22 (1943), No. 1.

8. Table of coefficients in numerical integration formulae, by A. N. Lowan and Hebert Salzer, J. Math. and Phys., vol. 22 (1943), No. 2.

9. Table of $Ji_0(x) = \int_x^\infty [J_0(t)/t]dt$ and related functions, by A. N. Lowan, G. Blanch, and M. Abramowitz, J. Math. and Phys., vol. 22 (1943), No. 2.

10. Table of Fourier coefficients, by A. N. Lowan and Jack Laderman, J. Math. and Phys., vol. 22 (1943), No. 3.

11. Coefficients for numerical differentiation with central differences, by H. E. Salzer, J. Math. and Phys., vol. 22 (1943), No. 3.

12. Seven-Point Lagrangian integration formulas, by G. Blanch and I. Rhodes, J. Math. and Phys., vol. 22 (1943), No. 4.

13. A short table of the first five zeros of the transcendental equation $J_0(x)Y_0(kx) - J_0(kx)Y_0(x) = 0$, by A. N. Lowan and A. Hillman, J. Math. and Phys., vol. 22 (1943), No. 4.

14. Table of coefficients for inverse interpolation with central differences, by H. E. Salzer, J. Math. and Phys., vol. 22 (1943), No. 4.

15. Table of $f_n(x) = [n!/(x/2)^n]J_n(x)$, by the Mathematical Tables Project, J. Math. and Phys., vol. 23 (1944), No. 1.

16. Table of coefficients for inverse interpolation with advancing differences, by H. E. Salzer, J. Math. and Phys., vol. 23 (1944), No. 2.

17. A new formula for inverse interpolation, by H. E. Salzer, Bull. Amer. Math. Soc., vol. 50 (1944), No. 8.

18. Coefficients for interpolation within a square grid in the complex plane, by A. N. Lowan and H. E. Salzer, J. Math. and Phys., vol. 23 (1944), No. 3.

19. Formulas for complex interpolation, by A. N. Lowan and H. E. Salzer, Quart. Appl. Math., vol. 2 (1944), No. 3.

20. Table of coefficients for differences in terms of the derivatives, by H. E. Salzer, J. Math. and Phys., vol. 23 (1944), No. 4.

21. On the distribution of errors in the nth tabular differences, by A. N. Lowan and J. Laderman, Ann. Math. Statist., vol. 10 (1939), No. 4.

22. Note on the computation of the differences of the $Si(x)$, $Ci(x)$, $Ei(x)$, and $Ei(-x)$ functions, by M. Abramowitz, Bull. Amer. Math. Soc., vol. 46 (1940), No. 4.

23. On the inversion of the q-series associated with Jacobian elliptic functions, by A. N. Lowan, G. Blanch, and W. Horenstein, Bull. Amer. Math. Soc., vol. 48 (1942), No. 10.

24. Roots of $\sin z = z$, by A. P. Hillman and H. E. Salzer, Philos. Mag., vol. 34 (ser. 7), August, 1943.

25. Coefficients for numerical integration with central differences, by H. E. Salzer, Philos. Mag., vol. 35 (Ser. 7), April, 1944.

THE MATHEMATICAL ASSOCIATION OF AMERICA

REPORT OF THE TREASURER FOR THE YEAR 1944

The following report of the Secretary-Treasurer as Treasurer for the year 1944 has been approved by the Finance Committee and accepted by vote of the Board of Governors.

I. Total Funds of the Association on December 31, 1943

(See Treasurer's report, pp. 177–180 of the Monthly for March, 1944)

Current Fund (checking account)		$ 2,262.47
Savings Account, Cleveland Trust Company		1,721.97
Savings Account, Oberlin Savings Bank		811.20
Invested Funds, Cleveland Trust Company		
Carus Fund	$ 7,016.73	
Chace Fund	8,974.07	
Houck Fund	8,436.24	
Chauvenet Fund	645.85	
Life Membership Fund	777.52	
General Fund	21,479.59	47,330.00
		$52,125.64

II. Current Fund Account for 1944

Receipts		Expenditures	
Balance, Jan. 1, 1944	$ 2,262.47	Monthly	
Individual dues	8,834.35	Publication	$ 4,922.05
Institutional dues	579.00	Reprints	51.44
Initiation fees	272.00	Editor-in-Chief's office	401.42
Subscriptions	2,726.54	Secretary-Treasurer's office	
Sale of back numbers Monthly	688.97	Clerical help	3,250.00
Advertisements	685.00	Postage	453.45
Sale of Archibald's Outline	57.00	Printing	247.29
Interest on General Fund	611.86	Office supplies	163.57
Interest on Carus Fund	199.91	Bank charges	79.33
Interest on Chace Fund	255.58	Exec. and Finance committees	116.68
Interest on Houck Fund	240.21	Regional Governors	161.56
Interest on Chauvenet Fund	18.22	Sections	89.35
Interest on Life Membership Fund	22.26	Subventions	
Interest from Hardy Fund	120.00	Amer. Math. Society	100.00
Sale of monographs (Carus)	1,136.99	Mathematical Reviews	500.00
Sale of Papyrus (Chace)	118.61	National Math. Magazine	200.00
From Oberlin Savings Bank	162.24	Back numbers Monthly	118.56
Miscellaneous sources	2.29	Register	578.56
Transferred from General Fund	1,941.78	Meetings	260.75
Transferred from Chauvenet Fund	31.78	Chauvenet prize (Chauv. Fund)	50.00
		Auditing	20.00
		Subscriptions to Annals	15.00
		B. F. Finkel (Hardy Fund)	120.00
		To General Fund	600.00
		To Ithaca Savings Bank	1,000.00
		Transferred to Carus Fund	1,336.90
		Transferred to Chace Fund	374.19
		Transferred to Houck Fund	240.21
		Transferred to Life Memb. Fund.	22.26
		Balance, Dec. 31, 1944	5,474.49
	$20,947.06		$20,947.06

III. Savings Account, Cleveland Trust Company

Balance, Jan. 1, 1944	$1,721.97		
Interest	12.55	Transferred to Invested Funds	$1,734.52
	$1,734.52		$1,734.52

IV. Savings Account, Ithaca Savings Bank

From Current Fund	$1,000.00	Balance, Dec. 31, 1944	$1,000.00

V. Invested Funds, Cleveland Trust Company

Market value of securities, Dec. 31, 1943	$47,330.00	Decrease in value of securities	$ 1.75
From Savings Account	1,734.52	Market value of securities, Dec. 30, 1944	49,622.00
From Current Fund	600.00	Cash balance, Dec. 31, 1944	40.77
	$49,664.52		$49,664.52

List of Securities

	Par Value	Market Value Dec. 30, 1944
U. S. Savings Bonds	$1,700.00	$ 1,610.00
U. S. Treasury Note, 1%, Ser. A, 1946	3,000.00	3,000.00
U. S. Treasury Bond, 2¾%, 1947	1,000.00	1,020.00
U. S. Treasury Bonds, 2%, 1950	3,000.00	3,060.00
U. S. Treasury Bonds, 1¼%, 1948	2,000.00	2,020.00
U. S. Savings Bonds, 2½%, Ser. G, 1953	3,000.00	2,850.00
U. S. Savings Bonds, 2½%, Ser. G, 1954	8,200.00	7,872.00
Canadian Nat. Ry. Co. Bonds, 4½%, 1956	2,000.00	2,300.00
Gatineau Power Co. 1st Mort. Bonds, 3¾%, Ser. A, 1969	2,000.00	2,080.00
Shawinigan W. and P. Co. 1st Mort. Bonds, 4¼%, 1970	2,000.00	2,060.00
C. and O. Ry. Co. Ref. Mort. Bonds, 3½%, Ser. D, 1996	3,000.00	3,150.00
Penn. R. R. Co. Gen. Mort. Bonds, 3½%, Ser. C, 1970	2,000.00	2,100.00
Union Pacific R. R. Co. Deb. Bonds	3,000.00	3,210.00
Amer. Tel. and Tel. Co. Conv. Deb. Bonds, 3%, 1956	2,000.00	2,460.00
Columbus and So. Ohio Elec. Co. 1st Mort. Bonds, 3¼%, 1970	2,000.00	2,180.00
Commonwealth Edison Co. Conv. Deb. Bonds, 3½%, 1958	2,000.00	2,340.00
Montana Power Co. 1st Ref. Mort. Bonds, 3¾%, 1966	3,000.00	3,150.00
New York Steam Corp. 1st Mort. Bond, 3½%, 1963	1,000.00	1,050.00
Texas Power and Light Co. 1st Mort. Bond, 5%, 1956	1,000.00	1,060.00
Phelps Dodge Corp. Conv. Deb. Bond, 3½%, 1952	1,000.00	1,050.00
		$49,622.00

VI. Carus Fund

Balance, Jan. 1, 1944	$7,016.73	Decrease in value of securities	$ .26
Sale of monographs	1,136.99		
Interest	199.91	Balance, Dec. 31, 1944	8,353.37
	$8,353.63		$8,353.63

VII. Chace Fund

Balance, Jan. 1, 1944	$8,974.07	Decrease in value of securities	$ .33
Sale of Papyrus	118.61		
Interest	255.58	Balance, Dec. 31, 1944	9,347.93
	$9,348.26		$9,348.26

VIII. Houck Fund

Balance, Jan. 1, 1944	$8,436.24	Decrease in value of securities	$ .31
Interest	240.21	Balance, Dec. 31, 1944	8,676.14
	$8,676.45		$8,676.45

IX. Chauvenet Fund

Balance, Jan. 1, 1944	$ 645.85	Decrease in value of securities	$ .03
Interest	18.22	Chauvenet prize	50.00
		Balance, Dec. 31, 1944	614.04
	$ 664.07		$ 664.07

X. Life Membership Fund

Balance, Jan. 1, 1944	$ 777.52	Decrease in value of securities	$ .03
Interest	22.26	Transferred to General Fund	44.72
		Balance, liability as of Dec. 31, 1944	755.03
	$ 799.78		$ 799.78

XI. General Fund (Invested)

Balance, Jan. 1, 1944	$21,479.59	Decrease in value of securities	$ .79
Transferred from Current Fund	600.00	Transferred to Current Fund	1,941.78
Transferred from Savings Acct., Cleveland Trust Company	1,734.52		
Transferred from Life Memb. Fund	44.72	Balance, Dec. 31, 1944	21,916.26
	$23,858.83		$23,858.83

XII. Total Funds of the Association, December 31, 1944

Current Fund (checking account)		$ 5,474.49
Savings Account (Ithaca Savings Bank)		1,000.00
Savings Account (Oberlin Savings Bank)		648.96
Invested Funds, Cleveland Trust Company		
Carus Fund	$ 8,353.37	
Chace Fund	9,347.93	
Houck Fund	8,676.14	
Chauvenet Fund	614.04	
Life Membership Fund	755.03	
General Fund	21,916.26	49,662.77
		$56,786.22

DECEMBER MEETING OF THE PHILADELPHIA SECTION

The annual meeting of the Philadelphia Section of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pa., on Saturday, December 2, 1944. Professor Anna Pell Wheeler, Chairman of the Section, presided at the morning and afternoon sessions.

There were twenty-seven present, including the following nineteen members of the Association: H. W. Brinkmann, P. A. Caris, S. F. Cavalli, J. W. Clawson, F. L. Dennis, Arnold Dresden, W. H. Gottschalk, V. H. Haag, Marguerite Lehr, F. L. Manning, A. E. Meder, Lillian Moore, F. D. Murnaghan, W. R. Murray, C. A. Nelson, C. O. Oakley, J. C. Oxtoby, A. D. Wallace, Anna Pell Wheeler.

At the business meeting the following officers were elected for the coming year: Chairman, C. A. Nelson, New Jersey College for Women, Rutgers University; Secretary, W. H. Gottschalk, University of Pennsylvania. The program committee for the next meeting will be F. L. Dennis, Ursinus College, J. C. Oxtoby, Bryn Mawr College (Chairman), and A. D. Wallace, University of Pennsylvania. The next meeting will be held at Philadelphia on December 1, 1945. At the instance of Professor Dresden, appropriate action was taken in memory of the late Professor James A. Shohat of the University of Pennsylvania.

The program consisted of the following papers:

1. *Mapping problems in aerial photography*, by Professor Marguerite Lehr, Bryn Mawr College.

This paper dealt with the grid method developed by the Canadian Topographical Survey for mapping unexplored land by means of oblique aerial photographs. Relation to the use of vertical photographs for surveying was outlined.

2. *Spherical triangles on a slide rule*, by Professor F. L. Dennis, Ursinus College.

The speaker derived an auxiliary law of sines which made possible the solution of all oblique spherical triangles on a specially designed slide rule. Several examples in navigation and astronomy were used to illustrate the reasonably high degree of accuracy of the method.

3. *Continuous flows and AP functions*, by Dr. W. H. Gottschalk, University of Pennsylvania.

A point x of a metric space in which there is defined a continuous flow is said to be almost periodic provided that to each $\epsilon>0$ there corresponds a relatively dense sequence $\{t_n\}$ of dates such that the distance between each x_{t_n} and x is less than ϵ. Let X denote the set of all bounded uniformly continuous complex-valued functions defined for all reals. Let X be metrized in the usual fashion, and let a continuous flow be defined in X by translation of the functions. Then a point x of X is almost periodic if and only if x is an almost peri-

odic function in the sense of Bohr. Bochner's characterization of almost periodic functions is then given an abstract topological setting. A point of a complete metric space in which there is defined a distance-preserving continuous flow is almost periodic if and only if its orbit is conditionally compact.

4. *The uniform tension of an elastic cylinder*, by Professor F. D. Murnaghan, Johns Hopkins University.

This paper will be printed in a subsequent issue of this MONTHLY.

P. M. WHITMAN, *Secretary*

THE FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Johns Hopkins University on Saturday, December 9, 1944. Professor C. H. Wheeler III, Chairman of the Section, presided at the morning and afternoon sessions.

There were fifty-two persons present, including the following thirty-three members of the Association: R. A. Baumgartner, T. E. Berry, S. G. Bourne, G. R. Clements, A. C. Cohen, Jr., A. T. Craig, A. E. Currier, J. A. Duerksen, B. H. Gere, Michael Goldberg, D. W. Hall, L. M. Kells, J. A. Larrivee, Florence P. Lewis, M. H. Martin, T. W. Moore, W. K. Morrill, F. D. Murnaghan, C. H. Rawlins, Irwin Roman, R. E. Root, E. D. Schell, A. D. Sollins, M. F. Smiley, C. V. L. Smith, J. H. Taylor, Marian M. Torrey, V. J. Varineau, R. W. Wagner, J. A. Ward, C. H. Wheeler, III, R. H. Wilson, Jr., Oscar Zariski.

It was decided to hold the spring meeting at George Washington University on either the first or second Saturday in May, 1945.

The morning program consisted of the following papers:

1. *Integral solutions of* $x^3+py^3=1$, p *a prime*, by Professor A. E. Currier, United States Naval Academy.

It was shown that this equation has a solution other than (1, 0) if and only if the fundamental unit of the ring 1, $3p$, $3p^2$ is of the form $x_1+y_1^3p$. Then (x_1, y_1) is the only such solution. A necessary condition is that $p=2$, or p be one of a finite set of primes of the form $17+18k$, or p be of one of the forms $1+18k$, $7+18k$.

2. *On the commutativity of certain rings*, by Professor N. Jacobson, Johns Hopkins University, introduced by Dr. W. K. Morrill.

3. *Configuration theorems*, by Dr. J. C. Abbott, United States Naval Academy, introduced by the Chairman.

This paper consisted of a partial classification of various configuration theorems according to their relative strength and rank, with particular emphasis on special cases of the Desargueian and Pappus theorems.

4. *Coördinates*, by T. J. Benac, United States Naval Academy, introduced by the Secretary.

Using the methods of introducing coördinates in the classical Desargueian plane, coördinates were introduced from alternative fields and Veblen-Wedderburn number systems in the non-Desargueian D_9 plane and the affine Little-Desargueian plane respectively. The concept of geomorphism was defined and illustrated, two algebraic systems being geomorphic if they define the same projective plane.

In the afternoon the Session heard an address by Professor Oscar Zariski on *The Riemann Manifolds of an Algebraic Function Field.*

W. K. MORRILL, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-Eighth Summer Meeting, Montreal, Canada, June 23–25, 1945.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Indianapolis, October 19, 1945
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Washington, D. C., May, 1945
METROPOLITAN NEW YORK, BROOKLYN, April 21, 1945
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 26, 1946
OHIO
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1945

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MAY 1945

The Summer Meeting of the Association to have been held at Montreal, June 23-25, 1945, has been cancelled at the request of the Office of Defense Transportation.

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